

A Banach space-valued ergodic theorem for amenable groups and applications

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Abstract

In this paper we study unimodular amenable groups. The first part is devoted to results on the existence of uniform families of ε -quasi tilings for these groups. In light of that, constructions of Ornstein and Weiss in [OW87] are extended by quantitative estimates for the covering properties of the corresponding decompositions. Afterwards, we apply the developed methods to obtain an abstract ergodic theorem for a class of functions mapping subsets of the group into some Banach space. Moreover, applications of this convergence result are studied: the uniform existence of the integrated density of states (IDS) for operators on amenable Cayley graphs; the uniform existence of the IDS for operators on discrete structures being quasi-isometric to some amenable group; the approximation of ℓ^2 -Betti numbers on cellular CW-complexes; the existence of certain densities of clusters in a percolated Cayley graph.

1 Introduction

In this paper we prove an ergodic theorem which is valid for all countable amenable groups. This statement applies to Banach space-valued functions which are defined on the space of all finite subsets of the group. Those mappings take their values according to a coloring \mathcal{C} of the group by finitely many colors (\mathcal{C} -invariance). As an ergodicity assumption, we deal with the case where the pattern frequencies induced by \mathcal{C} exist along the Følner sequence under consideration.

As a main example of those functions, we have eigenvalue counting functions of self-adjoint, finite hopping range operators in mind. In this situation, an appropriate coloring can for instance be induced by a random field determining some percolation process or the values of some Schrödinger potential.

The most important tools in the proof of the ergodic theorem are the results concerning ε -quasi tilings of the group, developed in the consecutive Sections 3 and 4. The challenge here is to approximate large compact sets in the group by unions of nearly disjoint translates of smaller Følner sets. In our elaborations, we make use of ideas and techniques of ORNSTEIN and WEISS elaborated in [OW87]. More precisely, we extend their results to obtain effective covering estimates. In light of that, we present the underlying constructions in a detailed and rigorous manner and we give a complete picture of the tilings under consideration.

These results enable us to prove the mentioned ergodic theorem in Section 5, cf. Theorem 5.5. Sufficient conditions for the validity of the ergodic theorem are discussed in Section

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6. In the following sections (7-10), we use this result to verify the (uniform) approximation of certain spectral quantities and distribution functions in various situations. The most important application will be the uniform existence of the integrated density of states (IDS) for finite hopping range, self-adjoint operators on Cayley graphs induced by arbitrary finitely generated, amenable groups. Further, we deal with operators on discrete structures which are quasi-isometric to some amenable group. Again, we confirm the uniform existence of the associated IDS. Moreover, we give a brief discussion of the approximation of ℓ^2 -Betti numbers on CW-complexes. We conclude our paper with a proof of the existence of certain densities associated with clusters in a percolated Cayley graph.

Having described the rough structure of the paper, let us now discuss its content in further detail. We also draw some connections between our work and the literature on similar topics. In Section 2 we introduce the most important notions for the groups under consideration. We outline the concepts of amenability, of the K -boundary and of different types of Følner sequences in the setting of unimodular groups. Further, some elementary properties are proven for these objects.

Section 3 is devoted to a first discussion of ε -quasi tilings of compact subsets of the group via ε -disjoint subsets. These methods are used to prove the main decomposition theorems in Section 4. Note that an ε -quasi tiling of a certain set T is a family of subsets such that the elements of this family may overlap only in a small portion and such that they cover all but a small fraction of T , c.f. Definition 4.1. We repeat some arguments of ORNSTEIN and WEISS to significantly extend them by useful quantitative estimates giving an exact description of the shape, as well as of the degree of uniformity of the approximation (tiling) of the set T . In light of that, Theorem 4.5 shows that each unimodular amenable group satisfies the so-called *special tiling property*. Roughly speaking - this condition assures the existence of ε -quasi tilings by translates of finitely many, arbitrarily invariant compact sets, such that the translates of one specific set cover a precisely determined portion. Considering families of such coverings in the case of countable groups, we show in Theorem 4.7 that on average, each part of the set T can be covered in the same way.

The covering arguments of Section 4 form the heart of the proof of the ergodic theorem, stated and proven in Section 5 as Theorem 5.5. So given an arbitrary countable amenable group G , along with a coloring \mathcal{C} of the group and with a Følner sequence (U_j) (along which the frequencies of all patterns exist) the ergodic theorem implies that for every almost additive and \mathcal{C} -invariant function F , mapping finite subsets of the group into some Banach space we have that

$$\overline{F} := \lim_{j \rightarrow \infty} \frac{F(U_j)}{|U_j|}$$

exists in the topology of the Banach space. Furthermore, we can express this limit using a semi-explicit formula, containing the frequencies of the patterns, as well as the densities of different tiles in an ε -quasi tiling, c.f. (ii) of Theorem 5.5. Similar ergodic theorems for discrete models in \mathbb{Z} or in the euclidean space can be found in [Len02, Kla07, LS06]. For results which are analogous to our ergodic theorem on certain restrictive geometries, see e.g. [LMV08, LSV10]. The hitherto most general result can be found in the latter paper, where the authors are able to deal with a specific class of amenable groups. More precisely, it is assumed that the group contains a Følner sequence consisting of monotiles for the group and the associated grids must be symmetric (cf. [LSV]). Although the validity of these conditions is satisfied for all residually finite, amenable groups, it is not clear at all how big the gap to the general case is, see also [OW87]. Using the sophisticated machinery of ε -quasi tiles, there

is no need to impose any restriction on the group, i.e. we establish the approximation result of Theorem 5.5 for *all* countable amenable groups. Considering the set \mathbb{R} of the real numbers as the Banach space, it is worth raising the issue of a similar convergence result for subadditive set functions f . Using the ε -quasi tilings proven in [OW87], it is shown in [Kri07, Kri10] that the limit $\lim_{j \rightarrow \infty} |U_j|^{-1} f(U_j)$ exists for Følner nets (U_j) . However, the author needs to assume periodicity of f (coloring of the group with just one color), a condition which is too restrictive for our purposes.

As our ergodic theorem applies to a large class of the functions F , we can deduce various approximation results for discrete models in a wide range of geometric situations. One major application is to show results concerning the approximability of the IDS of certain operators in Banach spaces. The roots of this topic can be found in the seminal papers [Pas71, Shu79]. From the view of spectral analysis, this is nowadays a well studied field. In light of that, a great variety of geometries and operators have been examined, both in a continuous setting [Szn89, Szn90, AS93, PV02, LPV04], as well as in a discrete setting [MY02, MSY03, DLM⁺03, Ves05]. All of the latter papers either demonstrate pointwise convergence of the approximants in all points on the real axis or even only in the continuity points of the limit function. Considering the fact that the set of discontinuity points of the IDS may be very large and even dense in the spectrum of the operator, c.f. [KLS03, Ves05, Sch12], pointwise convergence is much weaker than convergence with respect to the supremum norm. This observation triggered an increasing interest in models where uniform convergence of the approximating functions can be verified. As mentioned before, results in this direction have been obtained by using a Banach space-valued ergodic theorem. The idea is to define F as a map from finite subsets of the group to the eigenvalue counting function of the associated restricted operator. This strategy is known from earlier work, see e.g. [LS06, LMV08, LSV10] for the Banach space of bounded, right-continuous functions on \mathbb{R} , as well as [GLV11] for $L^p(I)$, where $I \subset \mathbb{R}$ is an arbitrary, bounded interval. Further, this method also allows for uniform approximation of the IDS on metric graphs possessing symmetries induced by some countable, amenable group, see e.g. [GLV07, PSS11].

The existence of frequencies of patterns along a Følner sequence is an assumption in the presented ergodic theorem. Section 6 provides sufficient conditions for the existence of frequencies in a randomly coloured graph.

In Section 7 of this paper, we consider the case where F stands for the eigenvalue counting function of some self-adjoint and finite hopping range operator on the Cayley graph of the group, restricted to the corresponding finite subset. With Theorem 5.5 at hand, we show the convergence along Følner sequences in the space of bounded, right-continuous functions endowed with the supremum norm. This directly generalizes the results concerning the IDS in [LSV10].

Another approach to IDS approximations is to deal with operators on discrete metric spaces which are quasi-isometric to some amenable group. Related results can e.g. be found in [Ele06] and in [LV09]. In Section 8, we use the abstract ergodic theorem to establish the uniform convergence of the eigenvalue counting functions of the restricted operators in the situation of a countable group. Thus, we can establish the Banach space convergence for a wide class of geometries. Dealing with one single finite hopping range, self-adjoint operator in a discrete setting instead of an ergodic family of operators, we can avoid the measure theoretical machinery used in [LV09].

Based on this result we show in the following Section 9 the approximation of ℓ^2 -Betti numbers for discrete spaces consisting of finite-dimensional cells, so-called CW-complexes. This

topic was studied in several papers with a related geometric setting, see [DM98, Eck99, Sch01]. Note that ℓ^2 -Betti numbers can be interpreted as the evaluation of the IDS at one single point. Therefore, it is not surprising that once one has shown uniform convergence, this implies results on the existence of these quantities.

In our last application (Section 10) we consider a bond percolation model for Cayley graphs. Here we pose the question whether certain densities of clusters of a fixed size are well defined. With the ergodic theorem at hand, we give a positive answer to that question and we prove the approximability of the associated distribution functions via finite volume analogues. Furthermore, we discuss the continuous dependency of these limits on the percolation parameter. As the distribution of cluster sizes in a percolation process is an intensively studied field, our results are complementary to other works. For instance, in [ADS80, BST10], the authors are interested in quantitative estimates concerning certain occurrence probabilities of clusters in the lattice case and for amenable groups, respectively. In particular, the asymptotic behaviour for large cluster sizes has been studied before in the literature, e.g. in connection with the so called “sharpness of the phase transition” [AN84, AB87, Men86, MMS88, AV10]

2 Preliminaries

Throughout this paper, we assume G to be a unimodular, second countable, amenable Hausdorff group.

The following section is devoted to the presentation of general properties of amenable groups. More precisely, we introduce a notion for a relative boundary of subsets in G and we use this concept to define so-called weak and strong Følner sequences. The existence of weak Følner sequences is commonly referred to as a characterization of amenability of second countable Hausdorff groups. As shown below, each strong Følner sequence is also a weak Følner sequence and the groups under consideration always possess a strong Følner sequence (see Lemma 2.6).

Let $\mathcal{B}(G)$ be the σ -algebra generated by the open sets of G . For a set $A \in \mathcal{B}(G)$ we denote by $|A|$ the Haar measure of A (for countable groups $|A|$ is the counting measure of A). The set of all finite subsets of G is called $\mathcal{F}(G)$. Independently of the Haar measure on G we write $\sharp(A)$ for the cardinality of a set $A \in \mathcal{F}(G)$. The unit element of the group is denoted by id .

We start with a definition of amenability.

Definition 2.1. An unimodular group G is called *amenable* if for each $\varepsilon > 0$ and $K \subseteq G$ compact there is a compact set $F \subseteq G$ and $K_0 \subseteq K$ such that

- (i) $|F \setminus kF| < \varepsilon|F|$ for all $k \in K_0$
- (ii) $|K \setminus K_0| < \varepsilon$

Let us proceed with definition of boundary terms of a group.

Definition 2.2. Let $\emptyset \neq K, T \subseteq G$ be compact subsets in G . We call the set $\partial_K(T)$, defined by

$$\partial_K(T) := \{g \in G \mid Kg \cap T \neq \emptyset \wedge Kg \cap (G \setminus T) \neq \emptyset\}$$

the K -boundary of the set T . Furthermore, T is called (K, δ) -invariant if

$$\frac{|\partial_K(T)|}{|T|} < \delta.$$

Remark 2.3. In section 7 and 10 we will consider the case where G is a finitely generated group, i.e. there exists a finite and symmetric generating system $S \subseteq G$. In this situation it is convenient to define the so called word metric $d_S : G \times G \rightarrow \mathbb{N}_0$ on G . Here one sets the distance of two distinct elements $x, y \in G$ to be smallest number of elements in S one needs to carry x into y , i.e.

$$d_S(x, y) := \min\{n \in \mathbb{N} \mid \exists s_1, \dots, s_n \in S \text{ with } s_1 \cdots s_n = xy^{-1}\} \quad \text{and} \quad d_S(x, x) := 0.$$

Using this distance one can define balls and boundaries as follows: the ball of radius $r \in \mathbb{N}$ around the element $x \in G$ is $B_r(x) := \{y \in G \mid d_S(x, y) \leq r\}$ and $B_r := B_r(\text{id})$. The r -boundary of a set $\Lambda \in \mathcal{F}(G)$ is

$$\partial^r(\Lambda) := \{x \in \Lambda \mid d_S(x, G \setminus \Lambda) \leq r\} \cup \{x \in G \setminus \Lambda \mid d_S(x, \Lambda) \leq r\}.$$

In this situation it is easy to show that $\partial_{B_r}(\Lambda) = \partial^r(\Lambda)$ holds for all $\Lambda \in \mathcal{F}(G)$ and $r \in \mathbb{N}$.

We will see below that the K -boundary Definition 2.2 can be used for an appropriate notion for a Følner condition for sets in amenable groups. In addition to that, it has very convenient properties which are easy to deal with. In the following lemma, we provide a short list of those properties which are important for our purposes.

Lemma 2.4. *Let $T, S, K \subseteq G$ be non-empty and compact and assume that $g \in G$. Then the following is true.*

- (i) $\partial_K(T) = \partial_K(G \setminus T)$.
- (ii) $\partial_K(S \cup T) \subseteq \partial_K(S) \cup \partial_K(T)$.
- (iii) $\partial_K(S \setminus T) \subseteq \partial_K(S) \cup \partial_K(T)$.
- (iv) $|\partial_K(S \setminus T)| \leq |\partial_K(T)| + |\partial_K(S)|$
- (v) $\partial_K(T) \subseteq \partial_L(T)$ if $K \subseteq L \subseteq G$.
- (vi) $\partial_K(Tg) = \partial_K(T)g$.
- (vii) $\partial_K(TS) \subseteq \partial_K(T)S$

Proof. The statements (i) to (vi) follow easily from Definition 2.2. To prove (vii) we fix $g \in \partial_K(TS)$. Thus $Kg \cap TS \neq \emptyset$ which implies that there is some $c \in S$ with $Kg \cap Tc \neq \emptyset$. Since $Kg \cap (G \setminus TS) \neq \emptyset$, it then follows that $Kg \cap (G \setminus Tc) \neq \emptyset$. We conclude that for every $g \in \partial_D(TS)$ we can find some $c \in S$ such that $gc^{-1} \in \partial_K(T)$, i.e. $g \in \partial_K(T)c \subseteq \partial_K(T)S$. ■

Definition 2.5. Let (F_n) be a sequence of non-empty compact subsets of a unimodular group G . If

$$\lim_{n \rightarrow \infty} \frac{|F_n \triangle KF_n|}{|F_n|} = 0$$

for all non-empty, compact $K \subseteq G$, then (F_n) is called *weak Følner sequence*. If

$$\lim_{n \rightarrow \infty} \frac{|\partial_K(F_n)|}{|F_n|} = 0$$

for all non-empty, compact $K \subseteq G$, then (F_n) is called *strong Følner sequence*. We say that a (weak or strong) Følner sequence (F_n) is *nested* if $\text{id} \in F_1$ and $F_n \subseteq F_{n+1}$ for all $n \geq 1$. Furthermore a (weak or strong) Følner sequence (F_n) is called *tempered* if there exists $c > 0$ such that

$$\left| \bigcup_{k < n} F_k^{-1} F_n \right| \leq c |F_n| \quad (n \in \mathbb{N}).$$

In [OW87] the authors proved that in each amenable group the following holds: given a compact set $K \subseteq G$ and $\delta > 0$ there is a compact set F which is (K, δ) -invariant. We will use this fact to prove that given an amenable, unimodular and second countable group one can always find strong Følner sequences. Furthermore, each strong Følner sequence is a weak Følner sequence.

Lemma 2.6. *Let G be unimodular. The following statements hold:*

- (a) *If G is amenable and second countable, then there exists a strong Følner sequence in G .*
- (b) *Each strong Følner sequence is a weak Følner sequence.*
- (c) *If there exists a weak Følner sequence in G , then G is amenable.*
- (d) *If G is countable, then each weak Følner sequence is also a strong Følner sequence.*
- (e) *If there exists a strong Følner sequence in G , then there exists also a nested strong Følner sequence in G .*
- (f) *Each (strong or weak) Følner sequence has a tempered subsequence.*

Proof. In order to prove (a) let G be amenable, second countable and unimodular. We denote by $\{V_n\}$ an enumeration of the countable base of the topology of G . Since G is locally compact, we can choose the V_n to be pre-compact. We now set $K_n := \bigcup_{j=1}^n \overline{V_j}$. Then each K_n is compact and we have $K_n \subseteq K_{n+1}$ for $n \geq 1$, as well as $\bigcup_n K_n = G$.

Let $K \subseteq G$ be compact. We claim that there is some $M \in \mathbb{N}$ such that $K \subseteq K_M$. For the proof, note first that for any $g \in G$, there is some $n(g) \in \mathbb{N}$ such that $g \in V_{n(g)}$. Hence the union $\bigcup_{g \in K} V_{n(g)}$ is an open cover of K . Since K is compact there must be a finite subcover $K \subseteq \bigcup_{j=1}^m V_{n(g_j)} \subseteq \bigcup_{j=1}^m \overline{V_{n(g_j)}}$. The latter union denotes a compact set and by construction of the K_n , we have $K \subseteq \bigcup_{j=1}^m \overline{V_{n(g_j)}} \subseteq K_M$, where $M := \max\{n(g_j) \mid j = 1, \dots, m\}$.

Take a sequence (ε_n) of positive numbers converging to 0. By the above mentioned statement in [OW87], we find for each $n \in \mathbb{N}$ a compact set F_n such that

$$\frac{|\partial_{K_n}(F_n)|}{|F_n|} < \varepsilon_n.$$

Hence, we conclude that for all $n \geq M$, one obtains with $K \subseteq K_M \subseteq K_n$ that

$$\frac{|\partial_K(F_n)|}{|F_n|} \leq \frac{|\partial_{K_M}(F_n)|}{|F_n|} \leq \frac{|\partial_{K_n}(F_n)|}{|F_n|} < \varepsilon_n.$$

So, clearly $\lim_{n \rightarrow \infty} |\partial_K(F_n)|/|F_n| = 0$.

Now assume that G is arbitrary and unimodular and (F_n) is some strong Følner sequence. To show (b) it is enough to verify that for each compact $F, K \subseteq G$ one has

$$F \triangle K F \subseteq \partial_{K \cup K^{-1} \cup \{\text{id}\}}(F),$$

since $K \cup K^{-1} \cup \{\text{id}\}$ is a compact set. To this end assume first that $g \in KF \setminus F$. Then $g = kf$ with $k \in K$ and $f \in F$, but $kf \notin F$. Define $L_K := K \cup K^{-1} \cup \{\text{id}\}$. Since $k^{-1} \in L_K$, one has $L_K g \cap F \neq \emptyset$. As $\text{id} \in L_K$, we derive $L_K g \cap (G \setminus F) \neq \emptyset$, hence $g \in \partial_{L_K}(F)$. Now assume that $h \in F \setminus KF$. Then for all $f \in F$ and all $k \in K$, $h \neq kf$, which implies $k^{-1}h \notin F$ for all $k \in K$. It follows from $K^{-1} \subseteq L_K$ that $L_K h \cap (G \setminus F) \neq \emptyset$. Since $\text{id} \in L_K$ and $h \in F$, we also have $L_K h \cap F \neq \emptyset$ and thus, $h \in \partial_{L_K}(F)$.

For part (c) of proof let (F_n) be a weak Følner sequence in a unimodular group G . Furthermore, let a compact $K \subseteq G$ and $\varepsilon > 0$ be given. As (F_n) is a weak Følner sequence, there exists some $n \in \mathbb{N}$ such that

$$\frac{|KF_n \setminus F_n|}{|F_n|} \leq \frac{|KF_n \triangle F_n|}{|F_n|} \leq \varepsilon.$$

As for each $k \in K$ we have $|F_n \setminus kF_n| = |kF_n \setminus F_n| \leq |KF_n \setminus F_n|$, the properties (i) and (ii) of Definition 2.1 hold with $K_0 = K$.

To show (d) we assume that G is countable and $|\cdot|$ denotes the counting measure. Let T and K be arbitrary compact sets and set $L_K := K \cup K^{-1} \cup \{\text{id}\}$. In this situation it is enough to prove

$$\partial_K(T) \subseteq \partial_{L_K}(T) \subseteq L_K(T \triangle L_K T),$$

since then $|\partial_K(T)| \leq |L_K| |T \triangle L_K T|$. Here the first inclusion holds by Lemma 2.4. To see the second inclusion, take some $g \in G$ such that $L_K g$ intersects non-trivially both T and $G \setminus T$. By the symmetry of L_K , we have $g \in L_K^{-1} T = L_K T$. If $g \notin T$, then $g \in L_K T \setminus T$ and since $\text{id} \in L_K$, we prove the claim for this case. If $g \in T$, find some $k \in L_K$ such that $kg \in L_K T \setminus T$, which exists since $L_K g \cap (G \setminus T)$ is non-empty. Again by the symmetry of L_K , we have $g \in L_K(L_K T \setminus T)$, which proves part (d) of the Lemma, as we have shown $g \in L_K(T \triangle L_K T)$ in both cases.

Now we prove (e). Let (F_n) be a strong Følner sequence in G . We choose some $x \in F_1$ and set $T_1 := F_1 x^{-1}$, then we proceed inductively. If T_1, \dots, T_k are chosen, then there is an $n \in \mathbb{N}$ such that F_n is $(T_k, 1)$ -invariant. As $\text{id} \in T_k$ we have $F_n \setminus \partial_{T_k}(F_n) \subseteq \{g \in F_n \mid T_k g \subseteq F_n\} =: S$ which gives with $|F_n \setminus \partial_{T_k}(F_n)| \geq |F_n| - |\partial_{T_k}(F_n)| > 0$ that S has positive Haar measure. Hence S is non-empty and we take some $g \in S$ and define $T_{k+1} := F_n g^{-1}$. This procedure gives a sequence (T_n) which is by construction nested and which is a strong Følner sequence as it is up to shifts a subsequence of (F_n) . Note that here we used unimodularity of G and (vi) of Lemma 2.4 which gives that shifts do not change the measure of the K -boundary.

Statement (f) was shown in [Lin01] for weak Følner sequences. By part (b) this holds for strong Følner sequences as well. ■

Remark 2.7. Lemma 2.6 shows that in an unimodular, second countable group amenability is equivalent to both the existence of a weak Følner sequence, as well as to the existence of a strong Følner sequence. For further discussions in that direction see for example [Gre69, OW87] and the seminal papers [Føl55, Ada93].

In the case of countable amenable groups we will only speak about Følner sequences as the specifications weak and strong are dispensable then.

In the following, for a number $b \in \mathbb{R}$, we will write $\lceil b \rceil$ for the smallest integer greater than or equal to b , i.e. $\lceil b \rceil := \inf\{m \in \mathbb{Z} \mid m \geq b\}$. Furthermore, we use the following notion. Given a number $0 < \varepsilon < 1$, the number $N(\varepsilon)$ is given by

$$N(\varepsilon) := \lceil \log(\varepsilon) / \log(1 - \varepsilon) \rceil. \quad (2.1)$$

For later considerations, we will need the following technical lemma.

Lemma 2.8. *Let $0 < \varepsilon < 1$ and assume that $(\alpha_i)_{i \in \mathbb{N}}$ is a complex-valued null sequence. Then,*

$$\lim_{\varepsilon \rightarrow 0} \sum_{i=1}^{N(\varepsilon)} \varepsilon(1 - \varepsilon)^{N(\varepsilon)-i} \cdot \alpha_i = 0.$$

Proof. Let $\delta > 0$ be arbitrary and choose $N_\delta \in \mathbb{N}$ such that $|\alpha_i| < \delta$ whenever $i \geq N_\delta$. As (α_i) is a null-sequence, there is some uniform bound $C > 0$ for the absolute values of α_i , $i \in \mathbb{N}$. We compute

$$\begin{aligned} \left| \sum_{i=1}^{N(\varepsilon)} \varepsilon(1 - \varepsilon)^{N(\varepsilon)-i} \alpha_i \right| &= \left| \sum_{i=1}^{N_\delta} \varepsilon(1 - \varepsilon)^{N(\varepsilon)-i} \alpha_i + \sum_{i=N_\delta+1}^{N(\varepsilon)} \varepsilon(1 - \varepsilon)^{N(\varepsilon)-i} \alpha_i \right| \\ &\leq C\varepsilon N_\delta + \delta \cdot \sum_{i=N_\delta+1}^{N(\varepsilon)} \varepsilon(1 - \varepsilon)^{N(\varepsilon)-i} \leq C\varepsilon N_\delta + \delta \end{aligned}$$

Here the last inequality holds since $\sum_{i=1}^{N(\varepsilon)} \varepsilon(1 - \varepsilon)^{N(\varepsilon)-i} = \sum_{i=0}^{N(\varepsilon)-1} \varepsilon(1 - \varepsilon)^i \leq 1$. Since δ is arbitrary, the claim follows. \blacksquare

3 Ornstein/Weiss tiling lemmas

This section is devoted to general tiling results for unimodular groups. We expand the elaborations of ORNSTEIN/WEISS in [OW87] to give quantitative estimates for coverings of subsets of the group, cf. Lemma 3.5. These results will be of great importance for the proofs of the tiling theorems in the next section.

For completeness reasons, we also include and prove two Lemmas (3.2 and 3.3) which have already been proven in the paper of ORNSTEIN/WEISS. Moreover, in Lemma 3.4, we give a rigorous proof of a statement that relies on a remark mentioned in [OW87].

We start with the notion of ε -disjoint subsets.

Definition 3.1. Let G be a group and assume that T_1, T_2 are subsets of G . For $0 < \varepsilon < 1$, we say that the sets T_1 and T_2 are ε -disjoint if there are sets $S_i \subseteq T_i$, $i = 1, 2$ such that

- (i) $S_1 \cap S_2 = \emptyset$,
- (ii) $|S_i| \geq (1 - \varepsilon)|T_i|$ for all $i = 1, 2$.

Let I be some index set. A family $\{T_i\}_{i \in I}$ of subsets of G is called ε -disjoint if any two sets of the family are ε -disjoint.

The following two Lemmas have already been proven in [OW87], Section I.3.

Lemma 3.2. *Let K be a non-empty, compact set in G containing the unit element id and let $T \subseteq G$ be (K, δ) -invariant. Then for the set*

$$S := \{g \in G \mid Kg \subseteq T\},$$

the following statements hold true:

$$(i) \quad |S| \geq (1 - \delta)|T|,$$

$$(ii) \quad \int_S \mathbf{1}_{Kc}(g) dc \leq |K| \text{ for all } g \in G.$$

Proof. Note first that $S = T \setminus \partial_K(T)$ such that by the fact that $\text{id} \in K$, (i) is satisfied. Further, consider translates Kc for $c \in S$. Then we derive the following formula which proves the second statement:

$$\int_S \mathbf{1}_{Kc}(g) dc = \int_S \mathbf{1}_K(gc^{-1})dc \leq \int_G \mathbf{1}_K(h) dh = |K|.$$

■

Lemma 3.3. *Let $K, S, T \subseteq G$ be non-empty compact sets, where $\text{id} \in K$, $|T| > 0$ and $|S|/|T| \geq 1 - \delta$ for some $0 < \delta < 1$. Then for all Borel sets $A \subseteq G$ with finite Haar measure there is some $c \in S$ such that*

$$|Kc \cap A| \leq \frac{|A| |K|}{|T|(1 - \delta)}. \quad (3.1)$$

Proof. Let A be a subset of G with finite Haar measure. Assume that for no $c \in S$ the Inequality (3.1) is satisfied. In this case we get

$$\int_S |Kc \cap A| dc > \int_S \frac{|A| |K|}{|T|(1 - \delta)} dc = \frac{|A| |K| |S|}{|T|(1 - \delta)} \geq |A| |K|, \quad (3.2)$$

where the last inequality is due to the fact that $|S|/|T| \geq 1 - \delta$. However, like in the proof of Lemma 3.2, we obtain

$$\int_S |Kc \cap A| dc = \int_S \int_A \mathbf{1}_{Kc}(g) dg dc = \int_A \int_S \mathbf{1}_K(gc^{-1}) dc dg \leq \int_A \int_G \mathbf{1}_K(h) dh dg = |A| |K|$$

which clearly is a contradiction to the strict Inequality (3.2). Thus, we find $c \in S$ such that (3.1) holds and our statement is proven. ■

We now prove the two main results of this section which will be exploited extensively in the next section. The first one states that given $\varepsilon > 0$ and given some (K, δ) -invariant set $T \subseteq G$, we can cover a portion of $\varepsilon(1 - 2\delta)$ of T by ε -disjoint translates of K . Actually, we assume K to be (B, ζ^2) -invariant for some non-empty, compact $B \subseteq G$. This fact will guarantee that there are pairwise disjoint subsets of the translates which are $(B, 4\zeta)$ -invariant and which still cover the same set as the translates of K .

To this end we will use the notion of maximal ε -disjointness. Let \mathcal{P} be a property which a subset of group G can obey, let I be some index set, $J \subseteq I$ and $\{K_i\}_{i \in I}$ a family of subsets of G . The family $\{K_i\}_{i \in J}$ is called *maximal ε -disjoint* with property \mathcal{P} , if $\{K_i\}_{i \in J}$ is ε -disjoint and each K_i satisfies \mathcal{P} , however for each $j \in I \setminus J$ such that K_j satisfies \mathcal{P} , the family $\{K_i\}_{i \in J \cup j}$ is no longer ε -disjoint. A family of *maximal disjoint* sets with property \mathcal{P} is defined analogously. In our examples the property \mathcal{P} will be “being a translate of a certain set” or/and “being a subset of a certain set”. We use for instance the term maximal ε -disjoint family of translates of K contained in T , where $K, T \subseteq G$.

Lemma 3.4. *Let G be some unimodular group, $0 < \varepsilon, \delta < 1/2$ and $0 < \zeta < \delta/4$. Furthermore let $T, K, B \subseteq G$ be compact sets such that T is (K, δ) -invariant, K is (B, ζ^2) -invariant and let the sets K and B contain the unit element. Then we can find finitely many elements $c_j, j = 1, \dots, n$ in T such that*

(i) $Kc_j \subseteq T, j = 1, \dots, n$.

(ii) $\{Kc_j\}_{j=1}^n$ is ε -disjoint.

(iii) for all $j = 1, \dots, n$, there is some set $K_j \subseteq K$ with $|K_j| \geq (1 - \varepsilon)|K|$ such that

- K_j is $(B, 4\zeta)$ -invariant
- $|\partial_B(K_j)| \leq |\partial_B(K)| + \zeta|K|$
- $\bigcup_{j=1}^n Kc_j = \bigcup_{j=1}^n K_jc_j$ and the latter union consists of pairwise disjoint sets.

(iv) $\left| \bigcup_{j=1}^n Kc_j \right| \geq \varepsilon(1 - 2\delta)|T|$.

Proof. We start the proof with the following claim: If $c_j \in T, j = 1, \dots, n$ are elements which fulfill conditions (i)-(iii) and

$$\left| \bigcup_{j=1}^n Kc_j \right| < \varepsilon(1 - 2\delta)|T|,$$

then there exists some $c_{n+1} \in T$ such that (i)-(iii) still hold for $c_j, j = 1, \dots, n+1$.

We postpone the proof of the claim and for the moment that it holds. Then we start with some maximal disjoint family $\{Kc_j\}_{j=1}^n$ of translates of K contained in T and set $K_j := K, j = 1, \dots, n$. Then obviously (i)-(iii) holds. If

$$\left| \bigcup_{j=1}^n Kc_j \right| \geq \varepsilon(1 - 2\delta)|T|,$$

then we are done with the proof. Otherwise we apply the claim and get some $c_{n+1} \in T$ such that conditions (i)-(iii) are still fulfilled for $c_j, j = 1, \dots, n+1$. Again, if condition (iv) is satisfied for c_1, \dots, c_{n+1} we are done, if not, we apply the claim again. This procedure will end after finitely many steps since T has finite measure and after each iteration we cover at least $(1 - \varepsilon)|K|$ more than before. Thus it remains to prove the claim.

Let $c_j \in T, j = 1, \dots, n$ be such that (i)-(iii) hold with sets $K_j, j = 1, \dots, n$ and $|A| < \varepsilon(1 - 2\delta)|T|$, where $A := \bigcup_{j=1}^n Kc_j$. We set $S := \{g \in T \mid Kg \subseteq T\}$ and

$$U := \left\{ g \in S \mid \frac{|Kg \cap \partial_B(A)|}{|K|} \leq \zeta \right\}.$$

By definition of U we have

$$\frac{|T \setminus U|}{|T|} \leq \frac{|T \setminus S|}{|T|} + \frac{|S \setminus U|}{|S|} \leq \delta + \int_S \frac{\mathbf{1}_{S \setminus U}(g)}{|S|} dg \leq \delta + \int_S \frac{|Kg \cap \partial_B(A)|}{\zeta|S||K|} dg$$

and furthermore, we use $|S| \geq (1 - \delta)|T|$, which follows from Lemma 3.2 part (i), Fubini's theorem and Lemma 3.2 part (ii) in the second inequality to obtain

$$\int_S \frac{|Kg \cap \partial_B(A)|}{\zeta|S||K|} dg \leq \frac{1}{\zeta(1 - \delta)|T||K|} \int_S \int_{\partial_B(A)} \mathbf{1}_{Kg}(h) dh dg \leq \frac{|\partial_B(A)|}{\zeta(1 - \delta)|T|}.$$

Clearly, the maximal number n of translates of K that can belong to A is bounded by $|T|/[(1-\varepsilon)|K|]$ such that we arrive at

$$\frac{|T \setminus U|}{|T|} \leq \delta + \frac{n|\partial_B(K)|}{\zeta(1-\delta)|T|} \leq \delta + \frac{|\partial_B(K)|}{\zeta(1-\delta)(1-\varepsilon)|K|} \leq \delta + \frac{\zeta}{(1-\varepsilon)(1-\delta)} \leq 2\delta,$$

where the last inequality follows from $\varepsilon, \delta < 1/2$ and $\zeta < \delta/4$. This yields $|U|/|T| \geq 1 - 2\delta$ which allows us to apply Lemma 3.3 to find some $c_{n+1} \in U$ such that

$$|Kc_{n+1} \cap A| \leq \frac{|A|}{|T|} \frac{1}{1-2\delta} |K| < \varepsilon |K|.$$

and hence condition (ii) holds for c_j , $j = 1, \dots, n+1$. As $c_{n+1} \in S$ we have $Kc_{n+1} \subseteq T$ which gives (i). We set $K_{n+1} := (Kc_{n+1} \setminus A)c_{n+1}^{-1}$ then by the above inequality we get $|K_{n+1}| \geq (1-\varepsilon)|K|$. To obtain the validity of the remaining part of the statement (iii) we make use of the inclusions which follow from elementary use of the definition of the B -boundary $\partial_B(\cdot)$

$$\begin{aligned} \partial_B(K \setminus Ac_{n+1}^{-1}) &\subseteq (\partial_B(K \setminus Ac_{n+1}^{-1}) \cap K) \cup (\partial_B(K \setminus Ac_{n+1}^{-1}) \cap G \setminus K) \\ &\subseteq ((\partial_B(K) \cup \partial_B(Ac_{n+1}^{-1})) \cap K) \cup \partial_B(K) \\ &\subseteq (\partial_B(Ac_{n+1}^{-1}) \cap K) \cup \partial_B(K). \end{aligned}$$

With $c_{n+1} \in U$, this yields

$$|\partial_B(K_{n+1})| \leq |\partial_B(K \setminus Ac_{n+1}^{-1})| \leq |K \cap \partial_B(Ac_{n+1}^{-1})| + |\partial_B(K)| \leq \zeta|K| + |\partial_B(K)|$$

and using $0 < \varepsilon < 1/2$, one obtains

$$\frac{|\partial_B(K_{n+1})|}{|K_{n+1}|} \leq \frac{\zeta|K|}{(1-\varepsilon)|K|} + \frac{|\partial_B(K)|}{(1-\varepsilon)|K|} \leq 2\zeta + 2\zeta^2 \leq 4\zeta.$$

Thus (iii) holds as well and the claim is proven. ■

Lemma 3.5. *Let G be some unimodular group, $0 < \varepsilon, \delta < 1/6$, $0 < \zeta < \delta/4$ and $\eta > 0$. Furthermore let $T, K, L, B \subseteq G$ be compact sets with $\text{id} \in L \subseteq K$, $\text{id} \in B$ and T be (KK^{-1}, δ) -invariant and K be (LL^{-1}, η) -invariant, as well as (B, ζ^2) -invariant. Then there is a set $C \in \mathcal{F}(G)$ such that $\{Kc\}_{c \in C}$ is ε -disjoint, $Kc \subseteq T$ for all $c \in C$, the set $T \setminus KC$ is $(LL^{-1}, 2\delta + \eta)$ -invariant and*

$$\varepsilon - \delta \leq \frac{|KC|}{|T|} \leq \varepsilon + \delta$$

holds. Furthermore, for each $c \in C$ there is a set $K_c \subseteq K$ which is $(B, 4\zeta)$ -invariant and satisfies $|K_c| \geq (1-\varepsilon)|K|$ and $|\partial_B(K_c)| \leq |\partial_B(K)| + \zeta|K|$. The sets $K_c c$, $c \in C$ are pairwise disjoint and $KC = \bigcup_{c \in C} K_c c$.

Proof. Exploiting property (v) of Lemma 2.4, we obtain that T is also (K, δ) -invariant and we infer from Lemma 3.4 that there is a finite set $\tilde{C} \subseteq \{g \in T \mid Kg \subseteq T\}$ such that the family $\{Kc\}_{c \in \tilde{C}}$ is ε -disjoint and $\varepsilon(1-2\delta)$ -covers T . Furthermore by Lemma 3.4 for each $c \in \tilde{C}$ there is a $(B, 4\zeta)$ -invariant set $K_c \subseteq K$ with $|K_c| \geq (1-\varepsilon)|K|$ and $|\partial_B(K_c)| \leq |\partial_B(K)| + \zeta|K|$ such that

$$K\tilde{C} = \bigcup_{c \in \tilde{C}} K_c c$$

and the sets $K_c, c \in \tilde{C}$ are pairwise disjoint. We note further that given some $g \in \partial_K(T)$, we have that $tg \in \partial_{KK^{-1}}(T)$ for all $t \in K$. Hence by the unimodularity of G , we derive $|K| \leq |\partial_{KK^{-1}}(T)|$. It follows that

$$\frac{|K|}{|T|} \leq \frac{|\partial_{KK^{-1}}(T)|}{|T|} < \delta.$$

This shows that if necessary, we can delete finitely many elements from \tilde{C} to obtain a subset C such that

$$\left| \frac{|KC|}{|T|} - \varepsilon \right| \leq \delta. \quad (3.3)$$

Applying the properties of the sets $K_c, c \in C$, inequality (3.3) implies

$$|T| \geq \frac{|KC|}{\varepsilon + \delta} = \frac{\sum_{c \in C} |K_c|}{\varepsilon + \delta} \geq \frac{1 - \varepsilon}{\varepsilon + \delta} |K| \cdot \#(C). \quad (3.4)$$

In light of that, using the upper bounds on ε and δ , we can compute

$$\frac{\#(C)}{|T| - |KC|} \leq \frac{\#(C)}{|T| - |K|\#(C)} \leq \frac{\#(C)}{(\frac{1-\varepsilon}{\varepsilon+\delta} - 1)|K|\#(C)} = \frac{\varepsilon + \delta}{(1 - 2\varepsilon - \delta)|K|} \leq \frac{1}{|K|}. \quad (3.5)$$

Now apply items (iv) and (vii) of Lemma 2.4 to obtain

$$\frac{|\partial_{LL^{-1}}(T \setminus KC)|}{|T \setminus KC|} \leq \frac{|\partial_{LL^{-1}}(T)|}{|T \setminus KC|} + \frac{|\partial_{LL^{-1}}(KC)|}{|T \setminus KC|} \leq \frac{|\partial_{LL^{-1}}(T)|}{|T| - |KC|} + \frac{\#(C) \cdot |\partial_{LL^{-1}}(K)|}{|T| - |KC|}. \quad (3.6)$$

It follows from Inequality (3.3) that $|T| - |KC| \geq (1 - (\varepsilon + \delta))|T|$. Combining this fact with Inequality (3.5), we deduce from (3.6) that

$$\frac{|\partial_{LL^{-1}}(T \setminus KC)|}{|T \setminus KC|} \leq \frac{|\partial_{LL^{-1}}(T)|}{(1 - (\varepsilon + \delta))|T|} + \frac{|\partial_{LL^{-1}}(K)|}{|K|} \leq 2\delta + \eta,$$

which shows our claim. Note that here, we used that T is (LL^{-1}, δ) -invariant since $L \subseteq K$ and since T is (KK^{-1}, δ) -invariant, c.f. Lemma 2.4, (v). \blacksquare

4 Tiling theorems

In this section we provide results concerning so-called ε -quasi-tilings of amenable groups. Continuing as before, the corresponding theorems are inspired by the work of ORNSTEIN and WEISS in [OW87], but also significantly extended by quantitative statements for the resulting tilings of invariant compact subsets of G . In light of that, we summarize the essential properties of such a quasi-tiling in Definition 4.1 and we show in Theorem 4.5 that these tilings exist in every unimodular amenable group. Though part of these properties have been shown by ORNSTEIN and WEISS, our theorem gives a far more detailed description of the quasi-tiling. More precisely, we are able to calculate exactly the densities determining the portion of covered mass by the translates of each individual tiling set T_i (see item (iv) of Definition 4.1). In part (b) of Theorem 4.5 we also show how one obtains a tiling with pairwise disjoint sets which can be made arbitrarily invariant with respect to any fixed compact subset

of G . While Theorem 4.5 proves the existence of one single ε -quasi-tiling, Theorem 4.7 deals with a family of ε -quasi-tilings and shows a uniform covering property. Roughly speaking, the latter statement says that among the different tilings, all admissible T_i -translates occur as a tile with the same density/probability (see Definition 4.6, property (II)).

As introduced in Equality (2.1), we will use $N(\varepsilon) = \lceil \log(\varepsilon)/\log(1-\varepsilon) \rceil$ for $0 < \varepsilon < 1$.

Definition 4.1 (Special tiling property). Let G be an amenable group. We say that G has the *special tiling property* if for any given β, ε with $0 < \beta < \varepsilon \leq 1/10$ and nested Følner sequence $\mathcal{S} = (S_n)$ there are sets

$$\{\text{id}\} \subseteq T_1 \subseteq T_2 \subseteq \cdots \subseteq T_{N(\varepsilon)} \quad \text{with} \quad T_i \in \{S_n \mid n \geq i\} \quad (1 \leq i \leq N(\varepsilon))$$

and some $\delta_0 = \delta(\beta) > 0$ depending only on β such that for all positive $\delta < \delta_0$ and every $(T_N T_N^{-1}, \delta)$ -invariant set $T \in \mathcal{B}(G)$, we can find so-called *center sets* $C_i^T \in \mathcal{F}(G)$, $i = 1, \dots, N(\varepsilon)$ such that

- (i) $T_i C_i^T \subseteq T$ for all $i = 1, \dots, N(\varepsilon)$,
- (ii) $\{T_i c\}_{c \in C_i^T}$ is an ε -disjoint family for all $i = 1, \dots, N(\varepsilon)$,
- (iii) $\{T_i C_i^T\}_{i=1}^{N(\varepsilon)}$ is a disjoint family of sets,
- (iv) $\left| \frac{|T_i C_i^T|}{|T|} - \eta_i(\varepsilon) \right| < \beta$ for all $i = 1, \dots, N(\varepsilon)$, where $\eta_i(\varepsilon) := \varepsilon(1-\varepsilon)^{N(\varepsilon)-i}$.

In this case, we say that the $\{T_i\}$ ε -quasi tile the group G and if for $T \subseteq G$, the properties (i)-(iv) hold, we say that T has the *special tiling property (STP)* with respect to $(\{T_i\}_{i=1}^{N(\varepsilon)}, \mathcal{S}, \varepsilon, \beta)$ and that T is ε -quasi tiled (with parameter β) by the basis sets T_i with finite center sets C_i^T .

Definition 4.2. We say that $S \subseteq G$ α -covers a set $T \subseteq G$ for $0 < \alpha \leq 1$ if

$$|S \cap T| \geq \alpha \cdot |T|.$$

Remark 4.3. Let us discuss the previous definitions. Item (iv) of the special tiling property shows that for $0 < \varepsilon \leq 1/10$ the values $\eta_i(\varepsilon) = \varepsilon(1-\varepsilon)^{N(\varepsilon)-i}$ can be interpreted as the “almost densities” of the elements T_i , $i \in \{1, \dots, N(\varepsilon)\}$ in the ε -quasi-tiling. This is emphasized by the fact that the $\eta_i(\varepsilon)$, $i \in \{1, \dots, N(\varepsilon)\}$ almost sum up to one (up to an ε). In fact we have

$$1 - \varepsilon \leq \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \leq 1.$$

This is clear as $N(\varepsilon) = \lceil \log(\varepsilon)/\log(1-\varepsilon) \rceil$ and

$$\sum_{i=1}^{N(\varepsilon)} \varepsilon(1-\varepsilon)^{N(\varepsilon)-i} = \varepsilon \sum_{i=0}^{N(\varepsilon)-1} (1-\varepsilon)^i = 1 - (1-\varepsilon)^{N(\varepsilon)} \leq 1. \quad (4.1)$$

Furthermore

$$1 - (1-\varepsilon)^{N(\varepsilon)} \geq 1 - (1-\varepsilon)^{\log(\varepsilon)/\log(1-\varepsilon)} = 1 - \varepsilon$$

holds for all $\varepsilon \in (0, 1)$.

Oftentimes we are interested in the case where $\beta = 2^{-N(\varepsilon)}\varepsilon$. Then it follows from Definition 4.1 that each $T \subseteq G$ with the STP is $(1 - 2\varepsilon)$ -covered by the corresponding T_i -translates. To see this, note that by the property (iv) and (4.1), we can compute

$$\frac{\left| \bigcup_{i=1}^{N(\varepsilon)} T_i C_i^T \cap T \right|}{|T|} = \sum_{i=1}^{N(\varepsilon)} \frac{|T_i C_i^T|}{|T|} \geq \sum_{i=1}^{N(\varepsilon)} \left(\varepsilon(1 - \varepsilon)^{N(\varepsilon)-i} - 2^{-N(\varepsilon)}\varepsilon \right) \geq 1 - (1 - \varepsilon)^{N(\varepsilon)} - \varepsilon$$

The definition of $N(\varepsilon)$ gives $(1 - \varepsilon)^{N(\varepsilon)} \leq \varepsilon$, which proves the claim.

Lemma 4.4. *Let G be a group and $T \in \mathcal{B}(G)$ satisfy the special tiling property with respect to $(\{T_i\}_{i=1}^{N(\varepsilon)}, \mathcal{S}, \varepsilon, \beta)$ and center sets C_i^T , $i = 1, \dots, N(\varepsilon)$. Then*

$$\sum_{i=1}^{N(\varepsilon)} \frac{|T_i| \#(C_i^T)}{|T|} \leq 2 \quad \text{and} \quad \left| \frac{\#(C_i^T)}{|T|} - \frac{\varepsilon(1 - \varepsilon)^{N(\varepsilon)-i}}{|T_i|} \right| < \frac{\beta}{|T_i|} + 2 \frac{\varepsilon \#(C_i^T)}{|T|}$$

for all $1 \leq i \leq N(\varepsilon)$.

Proof. It follows from the ε -disjointness of the $\{T_i c\}_{c \in C_i}$, $i = 1, \dots, N(\varepsilon)$ that

$$|T_i C_i^T| \leq |T_i| \#(C_i^T) = \sum_{c \in C_i^T} |T_i c| \leq \frac{1}{1 - \varepsilon} \sum_{c \in C_i^T} |T_i^{(c)} c| = \frac{1}{1 - \varepsilon} \left| \bigcup_{c \in C_i^T} T_i^{(c)} c \right| \leq \frac{1}{1 - \varepsilon} |T_i C_i^T|$$

where the $T_i^{(c)}$, $i = 1, \dots, N(\varepsilon)$, $c \in C_i^T$, are pairwise disjoint and $T_i^{(c)} \subseteq T_i$ with $|T_i^{(c)}| \geq (1 - \varepsilon) |T_i|$ for all $i = 1, \dots, N(\varepsilon)$, $c \in C_i^T$. Then since $\varepsilon < 1/2$,

$$\sum_{i=1}^{N(\varepsilon)} \frac{|T_i| \#(C_i^T)}{|T|} \leq \frac{\sum_{i=1}^{N(\varepsilon)} |T_i C_i^T|}{(1 - \varepsilon) |T|} = \frac{\left| \bigcup_{i=1}^{N(\varepsilon)} T_i C_i^T \right|}{(1 - \varepsilon) |T|} \leq 2,$$

which proves the first inequality. With a simple application of the triangle inequality, we prove the second claim. Namely,

$$\begin{aligned} \left| \frac{\#(C_i^T)}{|T|} - \frac{\varepsilon(1 - \varepsilon)^{N(\varepsilon)-i}}{|T_i|} \right| &= \frac{1}{|T_i|} \cdot \left| \frac{|T_i| \#(C_i^T)}{|T|} - \varepsilon(1 - \varepsilon)^{N(\varepsilon)-i} \right| \\ &\leq \frac{1}{|T_i|} \cdot \left| \frac{|T_i| \#(C_i^T)}{|T|} - \frac{|T_i C_i^T|}{|T|} \right| + \frac{1}{|T_i|} \cdot \left| \frac{|T_i C_i^T|}{|T|} - \varepsilon(1 - \varepsilon)^{N(\varepsilon)-i} \right| \\ &\leq \frac{\varepsilon}{1 - \varepsilon} \cdot \frac{|T_i| \#(C_i^T)}{|T_i| |T|} + \frac{1}{|T_i|} \cdot \left| \frac{|T_i C_i^T|}{|T|} - \varepsilon(1 - \varepsilon)^{N(\varepsilon)-i} \right| \end{aligned}$$

for all $1 \leq i \leq N$. The result follows from the property (iv) of Definition 4.1. ■

We now prove an extension of Theorem 6 in [OW87]. It shows that tilings of sufficiently invariant sets $T \subseteq G$ as described in Definition 4.1 always exist for unimodular amenable groups.

Theorem 4.5. *Let G be an unimodular amenable group. Then the following holds true.*

(a) *The group G satisfies the special tiling property.*

(b) Let β, ε and \mathcal{S} be given as in Definition 4.1 and let $B \subseteq G$ be compact with $\text{id} \in B$ and $0 < \zeta < \varepsilon$. Then we can find an ε -quasi-tiling with (B, ζ^2) -invariant sets T_i ($i = 1, \dots, N(\varepsilon)$) fulfilling all the properties of Definition 4.1 such that each $(T_N T_N^{-1}, \delta)$ -invariant set T with $0 < \delta < 6^{-N} \beta / 4$ can be tiled by translates $T_i c$, ($1 \leq i \leq N(\varepsilon)$, $c \in C_i^T$), which can be made disjoint in a way such that for all $1 \leq i \leq N(\varepsilon)$ and all $c \in C_i^T$, there is some set $T_i^{(c)} \subseteq T_i$ with

- $|T_i^{(c)}| \geq (1 - \varepsilon)|T_i|$,
- $T_i^{(c)}$ is $(B, 4\zeta)$ -invariant,
- $|\partial_B(T_i^{(c)})| \leq |\partial_B(T_i)| + \zeta |T_i|$,
- $T_i C_i^T = \bigcup_{c \in C_i^T} T_i^{(c)} c$, where the latter union consists of pairwise disjoint sets.

Proof. We will only prove (b) since (a) follows as a special case. Let ε and β with $0 < \beta < \varepsilon \leq 1/10$ and $\zeta > 0$ be given and let $\mathcal{S} = (S_n)$ be some nested Følner sequence. Choose some $0 < \delta < \frac{\beta}{2} 6^{-N(\varepsilon)}$. Without loss of generality we assume that S_n is (B, ζ^2) -invariant for all $n \in \mathbb{N}$ and that $\zeta < \delta/4$ (If ζ is not chosen to be smaller than $\delta/4$, then we can take some $\tilde{\zeta} < \delta/4$ and repeat all the steps of the proof again. Hence all claimed statements will hold for the original ζ as well). As usual, we use the notation $N := N(\varepsilon)$.

We start choosing the sets $T_i \in \{S_n \mid n \in \mathbb{N}\}$, $i = 1, \dots, N(\varepsilon)$ inductively in the following way: set $T_1 := S_1$ and if $T_i = S_k$ then take $T_{i+1} \in \{S_n \mid n \geq k+1\}$ which is $(T_i T_i^{-1}, \delta)$ -invariant. Then obviously $\text{id} \in T_i \subseteq T_{i+1}$ and $T_i \in \{S_n \mid n \geq i\}$ for all $i = 1, \dots, N(\varepsilon) - 1$.

Now assume that T is a $(T_N T_N^{-1}, \delta)$ -invariant subset of G . We apply Lemma 3.5 with $T = T$, $K = T_N$, $L = T_{N-1}$ and $B = B$ to obtain a finite set C_N^T such that $\{T_N c\}_{c \in C_N^T}$ consists of ε -disjoint subsets of T and $D_1 := T \setminus T_N C_N^T$ is $(T_{N-1} T_{N-1}^{-1}, \delta_1)$ -invariant, where $\delta_1 = 3\delta$. Furthermore there are $(B, 4\zeta)$ -invariant subsets $T_N^{(c)} \subseteq T_N$ such that $\bigcup_{c \in C_N^T} T_N^{(c)} c = \bigcup_{c \in C_N^T} T_N c$ as well as

$$|T_N^{(c)}| \geq (1 - \varepsilon)|T_N| \quad \text{and} \quad |\partial_B(T_N^{(c)})| \leq |\partial_B(T_N)| + \zeta |T_N| \quad \text{for all } c \in C_N^T.$$

Now we use Lemma 3.5 inductively. If for some $l \in \{1, \dots, N-1\}$ the set D_l is chosen as a $(T_{N-l} T_{N-l}^{-1}, \delta_l)$ -invariant set, we apply the Lemma with $T = D_l$, $K = T_{N-l}$, $L = T_{N-l-1}$, $B = B$, $\delta = \delta_l$, $\eta = \delta$ and $\zeta = \zeta$. Note that here it is important that δ_l is small enough, which we will ensure afterwards. This gives an appropriate set $C_{N-l}^T \in \mathcal{F}(G)$ such that $D_{l+1} := D_l \setminus T_{N-l} C_{N-l}^T$ is $(T_{N-l-1} T_{N-l-1}^{-1}, \delta_{l+1})$ -invariant, where $\delta_{l+1} := 2\delta_l + \delta$. Again there are $(B, 4\zeta)$ -invariant sets $T_{N-l}^{(c)} \subseteq T_{N-l}$ with $\bigcup_{c \in C_{N-l}^T} T_{N-l}^{(c)} c = \bigcup_{c \in C_{N-l}^T} T_{N-l} c$ as well as

$$|T_{N-l}^{(c)}| \geq (1 - \varepsilon)|T_{N-l}| \quad \text{and} \quad |\partial_B(T_{N-l}^{(c)})| \leq |\partial_B(T_{N-l})| + \zeta |T_{N-l}| \quad \text{for all } c \in C_{N-l}^T.$$

This will give the additional properties listed in (b).

With $\delta_0 = \delta$ we obtain $\delta_l = (2^{l+1} - 1)\delta$ for all $l = 1, \dots, N-1$. Therefore for arbitrary $l \in \{1, \dots, N-1\}$ we have $\delta_l \leq \delta_{N-1} = (2^N - 1)\delta \leq 1/6$, which shows that all δ_l are small enough to apply Lemma 3.5. Furthermore the Lemma implies the inequalities

$$\varepsilon - \delta_l \leq \frac{|T_{N-l} C_{N-l}^T|}{|D_l|} \leq \varepsilon + \delta_l \tag{4.2}$$

for all $l = 0, \dots, N-1$, where $D_0 := T$. We claim that for all $l = 0, \dots, N-1$, there is some constant κ_l independent of the parameters ε, β and δ such that

$$\left| \frac{|T_{N-l}C_{N-l}^T|}{|T|} - \varepsilon(1-\varepsilon)^l \right| \leq \kappa_l \cdot \delta_l. \quad (4.3)$$

We proceed by induction on l . Note that we have treated the case $l = 0$ in inequality (4.2) with $\kappa_0 = 1$. Now let $l \in \mathbb{N}$ and assume that (4.3) holds for all $k = 0, \dots, l-1$. By the induction hypothesis, we can sum up the resulting inequalities and arrive at

$$\varepsilon \cdot \sum_{k=0}^{l-1} (1-\varepsilon)^k - \sum_{k=0}^{l-1} \kappa_k \delta_k \leq \frac{|\bigcup_{k < l} T_{N-k} C_{N-k}^T|}{|T|} \leq \varepsilon \cdot \sum_{k=0}^{l-1} (1-\varepsilon)^k + \sum_{k=0}^{l-1} \kappa_k \delta_k.$$

Moreover, it follows from this and the definition of D_l that $T \setminus D_l = \bigcup_{k < l} T_{N-k} C_{N-k}^T$ and hence

$$1 - \varepsilon \cdot \sum_{k=0}^{l-1} (1-\varepsilon)^k - \sum_{k=0}^{l-1} \kappa_k \delta_k \leq \frac{|D_l|}{|T|} \leq 1 - \varepsilon \cdot \sum_{k=0}^{l-1} (1-\varepsilon)^k + \sum_{k=0}^{l-1} \kappa_k \delta_k. \quad (4.4)$$

Combining this inequality (4.4) with the estimate (4.2), we obtain

$$\begin{aligned} (\varepsilon - \delta_l) \left(1 - \varepsilon \sum_{k=0}^{l-1} (1-\varepsilon)^k - \sum_{k=0}^{l-1} \kappa_k \delta_k \right) &\leq \frac{|T_{N-l} C_{N-l}^T|}{|T|} \\ &\leq (\varepsilon + \delta_l) \left(1 - \varepsilon \sum_{k=0}^{l-1} (1-\varepsilon)^k + \sum_{k=0}^{l-1} \kappa_k \delta_k \right). \end{aligned}$$

It follows using $0 < \varepsilon < 1$ that

$$\begin{aligned} \varepsilon \left(1 - \varepsilon \sum_{k=0}^{l-1} (1-\varepsilon)^k \right) - \delta_l \left(1 + \sum_{k=0}^{l-1} \kappa_k \delta_k \right) - \sum_{k=0}^{l-1} \kappa_k \delta_k &\leq \frac{|T_{N-l} C_{N-l}^T|}{|T|} \\ &\leq \varepsilon \left(1 - \varepsilon \sum_{k=0}^{l-1} (1-\varepsilon)^k \right) + \delta_l \left(1 + \sum_{k=0}^{l-1} \kappa_k \delta_k \right) + \sum_{k=0}^{l-1} \kappa_k \delta_k, \end{aligned}$$

which implies with $\delta_k \leq \delta_l \leq 1$ and $1 - \varepsilon \sum_{k=0}^{l-1} (1-\varepsilon)^k = (1-\varepsilon)^l$ that

$$\left| \frac{|T_{N-l} C_{N-l}^T|}{|T|} - \varepsilon(1-\varepsilon)^l \right| \leq \delta_l \left(1 + \sum_{k=0}^{l-1} \kappa_k \delta_k \right) + \sum_{k=0}^{l-1} \kappa_k \delta_k \leq \left(1 + 2 \sum_{k=0}^{l-1} \kappa_k \right) \delta_l.$$

This shows the claim (4.3) with $\kappa_l := 1 + 2 \sum_{k=0}^{l-1} \kappa_k$. Since $\kappa_0 = 1$ we can compute the κ_l recursively, namely $\kappa_l = 3^l$ for all $l \geq 0$. In particular, we have for all $1 \leq i \leq N$ that

$$\left| \frac{|T_i C_i|}{|T|} - \varepsilon(1-\varepsilon)^{N-i} \right| \leq 3^N (2^{N+1} - 1) \delta \leq 2 \cdot 6^N \delta < \beta,$$

by the choice of δ . This proves (iv) of Definition 4.1. Properties (i), (ii) and (iii) follow by the construction of the sets C_k^T , $k = 1, \dots, N$. The additional properties concerning the disjoint sets $T_i^{(c)}$ can immediately be deduced from Lemma 3.5. \blacksquare

For countable amenable groups, we now verify the existence of a *family* of coverings of the type as in Theorem 4.5 which possesses a uniform covering property on average. To be precise, it will be shown that for each element u of the covered set T , the probability for this element being a center set of a covering of the family is equal to some number which only depends on ε as well as on the tiling set T_i .

To do so, we will need the concept of the so-called *uniform special tiling property*.

Definition 4.6. Let G be a countable amenable group. We say that G satisfies the *uniform special tiling property* (USTP) if for all strong Følner sequences $\{U_n\}$ in G , the following statements hold true.

- For given $0 < \varepsilon \leq 1/10$, $N := N(\varepsilon) := \lceil \log(\varepsilon)/\log(1 - \varepsilon) \rceil$, for arbitrary $0 < \beta < 2^{-N}\varepsilon$, and for any nested Følner sequence (S_n) , the group G satisfies the special tiling property according to Definition 4.1 with tiling sets

$$\{\text{id}\} \subseteq T_1 \subseteq T_2 \subseteq \cdots \subseteq T_{N(\varepsilon)},$$

where $T_i \in \{S_n \mid n \geq i\}$ for $1 \leq i \leq N(\varepsilon)$.

- For fixed numbers ε and β , there is some number $K \in \mathbb{N}$ depending on ε, β and the basis sets T_i such that for each $k \geq K$, we find a finite set $\Lambda_k \subseteq G$, along with a *family* $\{C_i^\lambda(U_k) \mid \lambda \in \Lambda_k, 1 \leq i \leq N\}$ of finite center sets for the T_i such that the set U_k is ε -quasi tiled with the properties (i)-(iii) of Definition 4.1 and

$$(I) \quad \frac{|\bigcup_{i=1}^N T_i C_i^\lambda(U_k)|}{|U_k|} \geq 1 - 4\varepsilon \text{ for all } \lambda \in \Lambda_k.$$

$$(II) \quad \left| |\Lambda_k|^{-1} \sum_{\lambda \in \Lambda_k} \mathbf{1}_{C_i^\lambda(U_k)}(u) - \frac{\varepsilon(1-\varepsilon)^{N-i}}{|T_i|} \right| < 3 \frac{\beta}{|T_i|} + \varepsilon \cdot \gamma_i \text{ for all } 1 \leq i \leq N \text{ and all } u \in U_k, \text{ where the } \gamma_i > 0 \text{ can be chosen such that } \sum_{i=1}^N \gamma_i |T_i| \leq 2.$$

Theorem 4.7 (Uniform Decompositions). *Each countable, amenable group satisfies the uniform special tiling property.*

Proof. Let $0 < \varepsilon \leq 1/10$ and $0 < \beta < 2^{-N}\varepsilon$ be fixed and let $\{U_n\}$ be a strong Følner sequence. Further, assume that $0 < \delta_0 < 6^{-N}\beta/16$.

Note that by Theorem 4.5, G is ε -quasi tiled by a finite sequence

$$\{\text{id}\} \subseteq T_1 \cdots \subseteq T_N$$

of compact basis sets taken from a nested Følner sequence (S_n) , where as usual, $N := N(\varepsilon) := \lceil \log(\varepsilon)/\log(1 - \varepsilon) \rceil$.

Let $0 < \varepsilon_1 < 1/100$. At various steps of the proof, we will have to make this parameter smaller. For the sake of the reader, we prefer doing this in a successive manner instead of imposing many technical conditions on ε_1 right now. This is possible since the restrictions will only depend on ε, β and the basis sets T_i , $1 \leq i \leq N$, but not on things developed in the proof. One can then think of starting the proof all over again with a new condition on the parameter ε_1 . We proceed in nine steps.

- (1) We let $M := \lceil \log(\varepsilon_1)/\log(1 - \varepsilon_1) \rceil$ and following Theorem 4.5, part (b), we find $(T_N T_N^{-1}, \delta_0^2)$ -invariant sets $\overline{T}_l \supseteq T_N$, $1 \leq l \leq M$, taken from a nested Følner sequence (S_n) , such that the \overline{T}_l ε_1 -quasi tile the group G . Then we can find some integer $K \in \mathbb{N}$ such that for each

$k \geq K$, the set $T := U_k$ is $(\overline{T}_l \overline{T}_l^{-1}, 2^{-l} \varepsilon_1)$ -invariant for all $1 \leq l \leq M$. Since ε_1 will depend on ε, β and the basis sets T_i , so does the integer number K . Further, we choose \hat{T} to be a (TT^{-1}, ε_1) -invariant compact set inheriting all the mentioned invariance properties of T . Using Theorem 4.5, we can also make sure that \hat{T} has the special tiling property with respect to $(\{\overline{T}_l\}_{l=1}^M, (S_n), \varepsilon_1, \beta_1)$, where $0 < \beta_1 < 2^{-M} \varepsilon_1$ (For instance, take $\hat{T} := U_{\tilde{K}}$ for $\tilde{K} \in \mathbb{N}$ large enough).

For $\overline{A} := \{a \in \hat{T} \mid TT^{-1}a \subseteq \hat{T}\}$, it follows from Lemma 3.2 that $|\overline{A}| \geq (1 - \varepsilon_1)|\hat{T}|$. Since $\overline{A} \subseteq sA$ with $A := \{g \in G \mid Tg \subseteq \hat{T}\}$ for every $s \in T$, we also have by unimodularity $|A| \geq (1 - \varepsilon_1)|\hat{T}|$.

- (2) Since \hat{T} has the special tiling property with respect to $(\{\overline{T}_l\}_{l=1}^M, (S_n), \varepsilon_1, \beta_1)$ ($0 < \beta_1 < 2^{-M} \varepsilon_1$), we can fix an ε_1 -tiling of \hat{T} as in Theorem 4.5, part (b), where we can make the \overline{T}_l -translates in this ε_1 -tiling actually disjoint such that the resulting disjoint translates $\overline{T}_l'(c) \subseteq \overline{T}_l c$ are still $(T_N T_N^{-1}, 4\delta_0)$ -invariant. We note that these disjoint translates $(1 - 2\varepsilon_1)$ -cover the set \hat{T} , i.e.

$$\begin{aligned} \frac{\left| \bigcup_{l=1}^M \bigcup_{c \in \overline{C}_l} \overline{T}_l'(c) \right|}{|\hat{T}|} &= \frac{\sum_{l=1}^M \sum_{c \in \overline{C}_l} |\overline{T}_l'(c)|}{|\hat{T}|} \\ &\geq (1 - 2\varepsilon_1). \end{aligned} \quad (4.5)$$

- (3) We have already mentioned that all the sets $\overline{T}_l'(c)$ must still be $(T_N T_N^{-1}, 4\delta_0)$ -invariant (recall that $4\delta_0 < 6^{-N} \beta / 4$). Therefore, by Theorem 4.5, we can fix in each translate $\overline{T}_l'(c)c$, ($c \in \overline{C}_l$) an ε -quasi tiling of $(T_i)_{i=1}^N$ with center sets $C_i(\overline{T}_l'(c)c)$ and

$$\left| \frac{|T_i C_i(\overline{T}_l'(c)c)|}{|\overline{T}_l'(c)c|} - \varepsilon(1 - \varepsilon)^{N-i} \right| < \beta \quad (4.6)$$

for $1 \leq i \leq N$. Further, we put

$$\hat{C}_i := \bigcup_{l=1}^M \bigcup_{c \in \overline{C}_l} C_i(\overline{T}_l'(c)c)$$

for $1 \leq i \leq N$ and note that the \hat{C}_i can be considered as center sets for the Følner elements T_i such that the $\{T_i c\}_{c \in \hat{C}_i}$ are ε -disjoint and such that for $1 \leq i < j \leq N$, the sets $T_i \hat{C}_i$ and $T_j \hat{C}_j$ are disjoint. For the covering properties of this ε -quasi-tiling, we compute with the fact that $\beta < 2^{-N} \varepsilon$ and with (4.6)

$$\begin{aligned} (1 - 2\varepsilon_1) |\hat{T}| &\stackrel{(4.5)}{\leq} \left| \bigcup_{l=1}^M \bigcup_{c \in \overline{C}_l} \overline{T}_l'(c) \right| = \sum_{l=1}^M \sum_{c \in \overline{C}_l} |\overline{T}_l'(c)| \\ &\stackrel{\beta < 2^{-N} \varepsilon}{\leq} (1 - 2\varepsilon)^{-1} \sum_{l=1}^M \sum_{c \in \overline{C}_l} \left| \bigcup_{i=1}^N T_i C_i(\overline{T}_l'(c)c) \right| \\ &= (1 - 2\varepsilon)^{-1} \left| \bigcup_{i=1}^N T_i \hat{C}_i \right| \end{aligned}$$

such that

$$\left| \bigcup_{i=1}^N T_i \hat{C}_i \right| \geq (1 - 2\varepsilon_1 - 2\varepsilon) |\hat{T}|. \quad (4.7)$$

- (4) We now would like to determine the portion of \hat{T} that is covered by each set $T_i \hat{C}_i$. We will see that up to some small error (2β), this will be $\varepsilon(1 - \varepsilon)^{N-i}$.

Note first that by the disjointness of the $\overline{T}_l'(c)c$ for all $c \in \overline{C}_l$ and all $1 \leq l \leq M$, we have

$$\begin{aligned} \frac{|T_i \hat{C}_i|}{|\hat{T}|} &= \frac{\sum_{l=1}^M \sum_{c \in \overline{C}_l} |T_i C_i(\overline{T}_l'(c)c)|}{|\hat{T}|} \\ &= \sum_{l=1}^M \sum_{c \in \overline{C}_l} \frac{|\overline{T}_l'(c)|}{|\hat{T}|} \frac{|T_i C_i(\overline{T}_l'(c)c)|}{|\overline{T}_l'(c)|}. \end{aligned} \quad (4.8)$$

On the one hand, the Inequality (4.6) yields

$$\begin{aligned} \frac{|T_i \hat{C}_i|}{|\hat{T}|} &\geq \sum_{l=1}^M \sum_{c \in \overline{C}_l} \frac{|\overline{T}_l'(c)|}{|\hat{T}|} (\varepsilon(1 - \varepsilon)^{N-i} - \beta) \\ &= \frac{\left| \bigcup_{l=1}^M \bigcup_{c \in \overline{C}_l} \overline{T}_l'(c)c \right|}{|\hat{T}|} (\varepsilon(1 - \varepsilon)^{N-i} - \beta) \\ &\stackrel{(4.5)}{\geq} (1 - 2\varepsilon_1)(\varepsilon(1 - \varepsilon)^{N-i} - \beta). \end{aligned} \quad (4.9)$$

On the other hand, starting from (4.8) again, we obtain

$$\begin{aligned} \frac{|T_i \hat{C}_i|}{|\hat{T}|} &\leq \sum_{l=1}^M \sum_{c \in \overline{C}_l} \frac{|\overline{T}_l'(c)|}{|\hat{T}|} (\varepsilon(1 - \varepsilon)^{N-i} + \beta) \\ &= \frac{\left| \bigcup_{l=1}^M \bigcup_{c \in \overline{C}_l} \overline{T}_l'(c)c \right|}{|\hat{T}|} (\varepsilon(1 - \varepsilon)^{N-i} + \beta) \\ &\leq \varepsilon(1 - \varepsilon)^{N-i} + \beta. \end{aligned} \quad (4.10)$$

Thus, by imposing a first condition on ε_1 (making it smaller if necessary), we deduce from the Inequalities (4.9) and (4.10) that

$$\left| \frac{|T_i \hat{C}_i|}{|\hat{T}|} - \varepsilon(1 - \varepsilon)^{N-i} \right| < 2\beta \quad (4.11)$$

for all $1 \leq i \leq N$.

- (5) We will see below that for each $1 \leq i \leq N$, the ratio $|\hat{C}_i|/|\hat{T}|$ plays an essential role in our argumentation. Hence, we compare this expression with the ratio $\varepsilon(1 - \varepsilon)^{N-i}/|T_i|$. By exploiting ε -disjointness of the T_i -translates, as well as (4.11), we obtain

$$\left| \frac{|\hat{C}_i|}{|\hat{T}|} - \frac{\varepsilon(1 - \varepsilon)^{N-i}}{|T_i|} \right| = \frac{1}{|T_i|} \left| \frac{|\hat{C}_i| |T_i|}{|\hat{T}|} - \varepsilon(1 - \varepsilon)^{N-i} \right|$$

$$\begin{aligned}
&\leq \frac{1}{|T_i|} \left| \frac{|\hat{C}_i| |T_i|}{|\hat{T}|} - \frac{|T_i \hat{C}_i|}{|\hat{T}|} \right| + \frac{1}{|T_i|} \left| \frac{|T_i \hat{C}_i|}{|\hat{T}|} - \varepsilon (1 - \varepsilon)^{N-i} \right| \\
&\leq \frac{1}{|T_i|} \cdot \varepsilon \cdot \frac{|\hat{C}_i| |T_i|}{|\hat{T}|} + \frac{2\beta}{|T_i|} \\
&= \varepsilon \frac{|\hat{C}_i|}{|\hat{T}|} + \frac{2\beta}{|T_i|} = \gamma_i \varepsilon + \frac{2\beta}{|T_i|},
\end{aligned} \tag{4.12}$$

where for $1 \leq i \leq N$, we put $\gamma_i := |\hat{C}_i|/|\hat{T}|$. Now Lemma 4.4 shows that as claimed in the statement of the theorem, we have $\sum_{i=1}^N \gamma_i |T_i| \leq 2$.

- (6) In the next step of the proof, it will be shown that most of the T -translates contained in \hat{T} will be $(1 - 3\varepsilon)$ -covered by the fixed pattern $\cup_{i=1}^N T_i \hat{C}_i$. Here, we will have to impose a second restriction on ε_1 . We recall from step (1) that we chose the set A as the collection of elements $a \in G$ such that the translate Ta lies entirely in \hat{T} and that

$$|A| \geq (1 - \varepsilon_1) |\hat{T}|. \tag{4.13}$$

For each $a \in A$, we set

$$\begin{aligned}
X(a) &:= \frac{|Ta \cap (\hat{T} \setminus \cup_{l=1}^M \cup_{c \in \bar{C}_l} \bar{T}_l'(c)c)|}{|Ta|} \\
&= \frac{|Ta \setminus \cup_{l=1}^M \cup_{c \in \bar{C}_l} \bar{T}_l'(c)c|}{|T|}
\end{aligned}$$

and treat X as a uniformly distributed (w.r.t. the Haar measure) random variable on the set A . It follows then from the Chebyshev Inequality that

$$\frac{|\{a \in A \mid X(a) > \sqrt{\varepsilon_1}\}|}{|A|} \leq \frac{1}{\sqrt{\varepsilon_1}} \sum_{a \in A} \frac{|Ta \setminus \cup_{l=1}^M \cup_{c \in \bar{C}_l} \bar{T}_l'(c)c|}{|A| \cdot |T|}.$$

Using (4.13) we further estimate by interchanging sums (Fubini's Theorem),

$$\begin{aligned}
&\leq \frac{1}{\sqrt{\varepsilon_1}} \sum_{a \in A} |A|^{-1} |T|^{-1} \sum_{g \in G} \mathbf{1}_{Ta \setminus (\cup_{l=1}^M \cup_{c \in \bar{C}_l} \bar{T}_l'(c)c)}(g) \\
&= \frac{1}{\sqrt{\varepsilon_1}} |A|^{-1} |T|^{-1} \sum_{a \in A} \sum_{g \in \hat{T} \setminus (\cup_{l=1}^M \cup_{c \in \bar{C}_l} \bar{T}_l'(c)c)} \mathbf{1}_{Ta}(g) \\
&= \frac{1}{\sqrt{\varepsilon_1}} |A|^{-1} |T|^{-1} \sum_{g \in \hat{T} \setminus (\cup_{l=1}^M \cup_{c \in \bar{C}_l} \bar{T}_l'(c)c)} \left(\sum_{a \in A} \mathbf{1}_{Ta}(g) \right) \\
&\leq \frac{1}{\sqrt{\varepsilon_1}} \frac{|\hat{T} \setminus \cup_{l=1}^M \cup_{c \in \bar{C}_l} \bar{T}_l'(c)c| \cdot |T|}{(1 - \varepsilon_1) |\hat{T}| \cdot |T|} \\
&\stackrel{4.5}{\leq} \frac{3\varepsilon_1 |\hat{T}|}{\sqrt{\varepsilon_1} (1 - \varepsilon_1) |\hat{T}|} = \frac{3\sqrt{\varepsilon_1}}{1 - \varepsilon_1} \stackrel{\varepsilon_1 < 1/2}{\leq} 6\sqrt{\varepsilon_1}.
\end{aligned} \tag{4.14}$$

This shows that for most of the a 's (up to a portion of $6\sqrt{\varepsilon_1}$), the corresponding translates Ta are $(1 - \sqrt{\varepsilon_1})$ -covered by the disjoint union

$$\bigcup_{l=1}^M \bigcup_{c \in \overline{C}_l, \overline{T}'_l(c) \cap Ta \neq \emptyset} \overline{T}'_l(c)c. \quad (4.15)$$

Recall that in step (1), we have chosen the set T as $(\overline{T}_l \overline{T}_l^{-1}, 2^{-l}\varepsilon_1)$ -invariant for all $1 \leq l \leq M$. Thus, since $\overline{T}_l c \subseteq \partial_{\overline{T}_l \overline{T}_l^{-1}}(Ta)$ for those $c \in \overline{C}_l$ ($1 \leq l \leq M$) with

$$\overline{T}'_l(c)c \cap Ta \neq \emptyset \wedge \overline{T}'_l(c)c \cap (G \setminus Ta) \neq \emptyset,$$

the total mass of all $\overline{T}'_l(c)$ -translates which make a contribution to the union (4.15) and which do not lie entirely in Ta divided by $|Ta| = |T|$ is bounded by

$$\sum_{l=1}^M \frac{|\partial_{\overline{T}_l \overline{T}_l^{-1}}(T)|}{|T|} \leq \sum_{l=1}^M 2^{-l} \varepsilon_1 \leq \varepsilon_1.$$

This implies that almost all translates Ta are $(1 - \sqrt{\varepsilon_1} - \varepsilon_1)$ -covered by translates $\overline{T}'_l(c)c$, $c \in \overline{C}_l$ that lie entirely in Ta . Since the translates $\overline{T}'_l(c)c$ are $(1 - 2\varepsilon)$ -covered by translates $T_i b$, $b \in \hat{C}_i$ (cf. Inequality (4.6) with $\beta < 2^{-N}\varepsilon$, Remark following Definition 4.1), we conclude that the union $\bigcup_{i=1}^N T_i \hat{C}_i$ covers at least a $(1 - 2\varepsilon)(1 - \sqrt{\varepsilon_1} - \varepsilon_1)$ -portion of Ta . To see this clearly, note that

$$\begin{aligned} \frac{|Ta \cap \bigcup_{i=1}^N T_i \hat{C}_i|}{|Ta|} &\geq (1 - 2\varepsilon) \frac{|Ta \cap \left(\bigcup_{l=1}^M \bigcup_{c \in \overline{C}_l, \overline{T}'_l(c)c \subseteq Ta} \overline{T}'_l(c)c \right)|}{|Ta|} \\ &\geq (1 - 2\varepsilon)(1 - \sqrt{\varepsilon_1} - \varepsilon_1). \end{aligned}$$

Next, by choosing ε_1 small enough (this is the second restriction on ε_1), we obtain that up to a portion of $6\sqrt{\varepsilon_1}$ of the elements $a \in A$, the translates Ta are $(1 - 3\varepsilon)$ -covered by the set $\bigcup_{i=1}^N T_i \hat{C}_i$.

- (7) We are now in position to define our family $\Lambda_k := \Lambda$ and the corresponding center sets C_i^λ for $1 \leq i \leq N$ and $\lambda \in \Lambda$. We obtain Λ by erasing from A the 'bad' elements, i.e.

$$\Lambda := \{\lambda \in A \mid X(\lambda) \leq \sqrt{\varepsilon_1}\}.$$

It follows from (4.13) and the considerations in the previous step that

$$|\Lambda| \geq (1 - 6\sqrt{\varepsilon_1})|A| \geq (1 - 6\sqrt{\varepsilon_1})(1 - \varepsilon_1) |\hat{T}|. \quad (4.16)$$

For $\lambda \in \Lambda$, we set

$$C_i^\lambda(T) := C_i^\lambda := \{d \in T \mid d\lambda \in \hat{C}_i\}.$$

for each $1 \leq i \leq N$. Then, for each $\lambda \in \Lambda$, the set T is in fact $(1 - 3\varepsilon)$ -covered by the expression $\bigcup_{i=1}^N T_i C_i^\lambda$. This can be seen as follows.

$$\frac{|T \cap \left(\bigcup_{i=1}^N T_i C_i^\lambda \right)|}{|T|} = \frac{|T \cap \left(\bigcup_{i=1}^N T_i (T \cap \hat{C}_i \lambda^{-1}) \right)|}{|T|}$$

$$\begin{aligned}
&= \frac{|T\lambda \cap \left(\bigcup_{i=1}^N T_i T\lambda\right) \cap \left(\bigcup_{i=1}^N T_i \hat{C}_i\right)|}{|T\lambda|} \\
&= \frac{|T\lambda \cap \bigcup_{i=1}^N T_i \hat{C}_i|}{|T\lambda|} \\
&\stackrel{\text{step(6)}}{\geq} 1 - 3\varepsilon.
\end{aligned}$$

Note that the statements (i)-(iii) follow from the construction of the C_i^λ . For the claimed $(1 - 4\varepsilon)$ -cover property, note that

$$T_i \cdot \{c \in C_i^\lambda \mid T_i c \cap T \neq \emptyset \wedge T_i c \cap (G \setminus T) \neq \emptyset\} \subseteq \partial_{T_i T_i^{-1}}(T)$$

for all $1 \leq i \leq N$. Since T is $(T_N T_N^{-1}, \varepsilon_1)$ -invariant with $\varepsilon_1 \ll 2^{-N} \varepsilon$ we get

$$\frac{\left| \bigcup_{i=1}^N \bigcup_{c \in C_i^\lambda(T), T_i c \cap (G \setminus T) \neq \emptyset} T_i c \right|}{|T|} \leq \sum_{i=1}^N \frac{|\partial_{T_i T_i^{-1}}(T)|}{|T|} \leq \varepsilon,$$

which shows property (I) of Definition 4.6.

We still need to show the uniform covering principle (vi).

- (8) Recall that in step (5), we had a closer look at the ratios $\gamma_i = |\hat{C}_i|/|\hat{T}|$ which we compared to the expression $\varepsilon(1 - \varepsilon)^{N-1}/|T_i|$ for all $1 \leq i \leq N$. We will show now that we can choose ε_1 small enough (third and fourth restriction on ε_1) such that

$$\left| |\Lambda|^{-1} \sum_{\lambda \in \Lambda} \mathbf{1}_{C_i^\lambda}(u) - \frac{|\hat{C}_i|}{|\hat{T}|} \right| < \frac{\beta}{|T_i|} \quad \text{for all } u \in T. \quad (4.17)$$

At first, we note that

$$\begin{aligned}
u \in C_i^\lambda &\iff u \in T \wedge u\lambda \in \hat{C}_i \\
&\iff u \in T \wedge \lambda \in u^{-1}\hat{C}_i
\end{aligned}$$

and thus, we obtain

$$|\Lambda|^{-1} \sum_{\lambda \in \Lambda} \mathbf{1}_{C_i^\lambda}(u) = |\Lambda|^{-1} \sum_{\lambda \in \Lambda} \mathbf{1}_{u^{-1}\hat{C}_i}(\lambda) = \frac{|\Lambda \cap u^{-1}\hat{C}_i|}{|\Lambda|} \quad (4.18)$$

for $1 \leq i \leq N$ and $u \in T$. We now compare the right hand side of equality (4.18) with the γ_i .

On the one hand, we obtain with (4.16)

$$\begin{aligned}
\frac{|\Lambda \cap u^{-1}\hat{C}_i|}{|\Lambda|} &= \frac{|u\Lambda \cap \hat{C}_i|}{|\Lambda|} \\
&\leq \frac{|\hat{T} \cap \hat{C}_i|}{(1 - 6\sqrt{\varepsilon_1})(1 - \varepsilon_1)|\hat{T}|}
\end{aligned}$$

$$= \frac{|\hat{C}_i|}{(1 - 6\sqrt{\varepsilon_1})(1 - \varepsilon_1)|\hat{T}|},$$

such that with $\gamma_i \leq 1$ (in fact $\hat{C}_i \subseteq \hat{T}$)

$$\frac{|\Lambda \cap u^{-1}\hat{C}_i|}{|\Lambda|} - \frac{|\hat{C}_i|}{|\hat{T}|} \leq \left(\frac{1}{(1 - 6\sqrt{\varepsilon_1})(1 - \varepsilon_1)} - 1 \right).$$

for all $1 \leq i \leq N$. By making ε_1 small enough (third restriction), this simplifies to

$$\frac{|\Lambda \cap u^{-1}\hat{C}_i|}{|\Lambda|} - \frac{|\hat{C}_i|}{|\hat{T}|} \leq \frac{\beta}{|T_i|} \quad (4.19)$$

for all $1 \leq i \leq N$ and all $u \in T$. On the other hand,

$$\begin{aligned} \frac{|\Lambda \cap u^{-1}\hat{C}_i|}{|\Lambda|} &\geq \frac{|u\Lambda \cap \hat{C}_i|}{|\hat{T}|} \geq \frac{|\hat{T} \cap \hat{C}_i|}{|\hat{T}|} - \frac{|\hat{T} \setminus u\Lambda|}{|\hat{T}|} \\ &= \frac{|\hat{C}_i|}{|\hat{T}|} - \frac{|\hat{T}| - |u\Lambda|}{|\hat{T}|} \\ &= \frac{|\hat{C}_i|}{|\hat{T}|} - \frac{|\hat{T}| - |\Lambda|}{|\hat{T}|} \\ &\geq \frac{|\hat{C}_i|}{|\hat{T}|} - [1 - (1 - 6\sqrt{\varepsilon_1})(1 - \varepsilon_1)] \end{aligned} \quad (4.20)$$

Therefore, with a fourth (and a last) restriction on ε_1 , we finally obtain

$$\frac{|\hat{C}_i|}{|\hat{T}|} - \frac{|\Lambda \cap u^{-1}\hat{C}_i|}{|\Lambda|} \leq \frac{\beta}{|T_i|} \quad (4.21)$$

for all $1 \leq i \leq N$ and all $u \in T$.

Combining the Inequalities (4.19) and (4.21) with Equality (4.18), we arrive at the claim (4.17).

- (9) Finally, we are able to prove claim (II) of the statement. We combine the Inequalities (4.12) and (4.17) and yield by means of the triangle inequality

$$\begin{aligned} &\left| |\Lambda|^{-1} \sum_{\lambda \in \Lambda} \mathbf{1}_{C_i^\lambda}(u) - \frac{\varepsilon(1 - \varepsilon)^{N-i}}{|T_i|} \right| \\ &\leq \left| |\Lambda|^{-1} \sum_{\lambda \in \Lambda} \mathbf{1}_{C_i^\lambda}(u) - \frac{|\hat{C}_i|}{|\hat{T}|} \right| + \left| \frac{|\hat{C}_i|}{|\hat{T}|} - \frac{\varepsilon(1 - \varepsilon)^{N-i}}{|T_i|} \right| \\ &\leq \frac{\beta}{|T_i|} + \gamma_i \cdot \varepsilon + \frac{2\beta}{|T_i|} \\ &= 3 \frac{\beta}{|T_i|} + \gamma_i \cdot \varepsilon \end{aligned}$$

for all $1 \leq i \leq N$ and all $u \in T$, where $\sum_{i=1}^N \gamma_i |T_i| \leq 2$.

So, we have finally finished the proof of the theorem. ■

We include a useful consequence.

Corollary 4.8. *Assume that G is a countable, amenable group. Let some uniform family $\Lambda_k \subseteq G$ of ε -quasi tilings of some set U_k be fixed as in Definition 4.6 with parameters $0 < \varepsilon \leq 1/10$, $0 < \beta < 2^{-N(\varepsilon)}\varepsilon$ and basis sets $\{T_i\}_{i=1}^{N(\varepsilon)}$. Then, if for each $\lambda \in \Lambda_k$, we consider the counting function*

$$Z^\lambda : G \rightarrow \mathbb{N}_0 : Z^\lambda(g) := \sum_{i=1}^{N(\varepsilon)} \sum_{c \in C_i^\lambda} \mathbf{1}_{T_i c}(g),$$

we obtain that

$$\sup_{g \in G} \left(|\Lambda_k|^{-1} \sum_{\lambda \in \Lambda_k} Z^\lambda(g) \right) < 2\varepsilon + 1.$$

Proof. Fix $g \in G$. Then

$$\begin{aligned} \left(|\Lambda_k|^{-1} \sum_{\lambda \in \Lambda_k} Z^\lambda(g) \right) &= |\Lambda_k|^{-1} \sum_{\lambda \in \Lambda_k} \sum_{i=1}^{N(\varepsilon)} \sum_{c \in C_i^\lambda} \mathbf{1}_{T_i c}(g) \\ &= \sum_{i=1}^{N(\varepsilon)} |\Lambda_k|^{-1} \sum_{\lambda \in \Lambda_k} |T_i^{-1}g \cap C_i^\lambda|. \end{aligned}$$

Now, by Property (II) of Definition 4.6, we obtain by summation over $T_i^{-1}g$ and by the means of the triangle inequality that

$$\begin{aligned} \left(|\Lambda_k|^{-1} \sum_{\lambda \in \Lambda_k} Z^\lambda(g) \right) &\leq \sum_{i=1}^{N(\varepsilon)} \left(\beta + \varepsilon(1 - \varepsilon)^{N(\varepsilon)-i} \right) \\ &\leq 2\varepsilon + 1, \end{aligned}$$

since $\beta < 2^{-N(\varepsilon)}\varepsilon$. As $g \in G$ was arbitrary, this finishes the proof. ■

Remark 4.9. The functions Z^λ measure the overlap for one element $g \in G$ in the ε -quasi tiling with center sets C_i^λ , ($\lambda \in \Lambda_k, 1 \leq i \leq N(\varepsilon)$). Corollary 4.8 shows that on average over the family Λ_k , each element in G is contained in *less* than two different translates of the basis sets T_i , ($1 \leq i \leq N(\varepsilon)$).

5 An almost additive Ergodic Theorem

Let G be a discrete, countable amenable group and denote by $\mathcal{F}(G)$ the set of compact (finite) subsets in G . For the following elaborations, we refer to the setting in [LSV10]. In their work, the authors prove a Banach space valued ergodic theorem for functions $F : \mathcal{F}(G) \rightarrow (X, \|\cdot\|)$ with certain boundedness and additivity conditions (Theorem 3.1). However, it has been necessary to impose strong restrictions on the group G under consideration. More precisely, one has to assume that G possesses a Følner sequence $\{Q_n\}$ with the property that each element Q_n is a monotile of G , i.e. for each $n \in \mathbb{N}$ there is a grid $G_n \subseteq G$ such that $\cup_{g \in G_n} Q_n g$

is a disjoint tiling of the group. In addition to that it turned out that this assumption is even not sufficient, but one also has to require the grid sets to be symmetric (cf. [LSV]). Using the ε -quasi tilings of the previous section, we can drop all these restrictions to prove a general Banach valued ergodic theorem (cf. Theorem 5.5).

Assume that we are given a finite set \mathcal{A} of colors. Then each map $\mathcal{C} : G \rightarrow \mathcal{A}$ defines a coloring of the group. If $\mathcal{F}(G)$ denotes the set of all finite subsets of G , then we call a map

$$P : D(P) \rightarrow \mathcal{A}$$

a *pattern* with $D(P) \in \mathcal{F}(G)$ as the *domain* of P . The set of all patterns is denoted by \mathcal{P} and for a fixed $Q \in \mathcal{F}(G)$ the subset of \mathcal{P} which contains only the patterns with domain Q is denoted by $\mathcal{P}(Q)$. Given a set $Q \subseteq D(P)$ and an element $x \in G$, we furthermore define a *restriction of a pattern* by

$$P|_Q : Q \rightarrow \mathcal{A} : g \mapsto P|_Q(g) = P(g),$$

as well as a *translation of a pattern* by

$$Px : D(P)x \rightarrow \mathcal{A} : yx \mapsto P(y).$$

Translations and restrictions of colorings are defined equivalently. Two patterns are called *equivalent* if one is the translation of the other. The equivalence class of a pattern P is then denoted by \tilde{P} . We write $\tilde{\mathcal{P}}$ for the induced set of equivalence classes in \mathcal{P} . For two patterns P and P' , the *number of occurrences* of the pattern P in P' is denoted by

$$\#_P(P') := \left| \{x \in G \mid D(P)x \subseteq D(P'), P'|_{D(P)x} = Px\} \right|.$$

Counting occurrences of patterns along a Følner sequence $(U_j)_{j \in \mathbb{N}}$ leads to the definition of *frequencies*. If for a pattern P and a Følner sequence $(U_j)_{j \in \mathbb{N}}$ the limit

$$\nu_P := \lim_{j \rightarrow \infty} \frac{\#_P(\mathcal{C}|_{U_j})}{|U_j|}$$

exists, we call ν_P the *frequency of P in the coloring \mathcal{C} along $(U_j)_{j \in \mathbb{N}}$* .

Definition 5.1 (boundary term). A function $b : \mathcal{F}(G) \rightarrow [0, \infty)$ is called a *boundary term* if

- (i) $b(Q) = b(Qx)$ for all $x \in G$ and all $Q \in \mathcal{F}(G)$.
- (ii) $\lim_{j \rightarrow \infty} \frac{b(U_j)}{|U_j|} = 0$ for any Følner sequence $(U_j)_{j \in \mathbb{N}}$.
- (iii) there exists $D > 0$ with $b(Q) \leq D|Q|$ for all $Q \in \mathcal{F}(G)$.
- (iv) one has for all $Q, Q' \in \mathcal{F}(G)$

$$b(Q \cap Q') \leq b(Q) + b(Q'), \quad b(Q \cup Q') \leq b(Q) + b(Q'), \quad b(Q \setminus Q') \leq b(Q) + b(Q').$$

For a pattern P we define $b(P) := b(D(P))$. Note that due to property (i), the value $b(P)$ depends only on the equivalence class of a pattern.

Definition 5.2. Let a Banach-space $(X, \|\cdot\|)$, a finite set \mathcal{A} , a coloring $\mathcal{C} : G \rightarrow \mathcal{A}$ and a function $F : \mathcal{F}(G) \rightarrow X$ be given.

- F is called *almost additive* if there is boundary term $b : \mathcal{F}(G) \rightarrow [0, \infty)$ such that for any disjoint $Q_1, \dots, Q_k \in \mathcal{F}(G)$ one has

$$\left\| F(Q) - \sum_{i=1}^k F(Q_i) \right\| \leq \sum_{i=1}^k b(Q_i),$$

where $Q = \bigcup_{i=1}^k Q_i$.

- F is called \mathcal{C} -invariant if for any $Q, U \in \mathcal{F}(G)$ the equivalence of the patterns $\mathcal{C}|_Q$ and $\mathcal{C}|_U$ implies $F(Q) = F(U)$.

Given an almost additive and \mathcal{C} -invariant function $F : \mathcal{F}(G) \rightarrow X$ we define $\tilde{F} : \mathcal{P} \rightarrow X$ by

$$\tilde{F}(P) = \begin{cases} F(Q) & \text{if } Q \in \mathcal{F}(G) \text{ such that } \mathcal{C}|_Q = \tilde{P} \\ 0 & \text{else.} \end{cases} \quad (5.1)$$

This is well defined by the \mathcal{C} -invariance of F . The next result gives properties of the functions F and \tilde{F} .

Lemma 5.3. *Let a Banach space $(X, \|\cdot\|)$, a finite set \mathcal{A} and a coloring $\mathcal{C} : G \rightarrow \mathcal{A}$ be given. Furthermore let $F : \mathcal{F}(G) \rightarrow X$ be \mathcal{C} -invariant and almost additive with boundary term b .*

- (i) *Then there exists a constant $C > 0$ such that*

$$\|F(Q)\| \leq C|Q| \quad \text{and} \quad \|\tilde{F}(P)\| \leq C|D(P)|,$$

for all $Q \in \mathcal{F}(G)$ and $P \in \mathcal{P}$, where \tilde{F} is given by (5.1).

- (ii) *If furthermore $0 < \varepsilon < 1/2$ is given and Q_i , $i = 1, \dots, k$ are ε -disjoint sets and $Q = \bigcup_{i=1}^k Q_i$, then*

$$\left\| F(Q) - \sum_{i=1}^k F(Q_i) \right\| \leq (3C + 9D)\varepsilon|Q| + 3 \sum_{i=1}^k b(Q_i),$$

where C is the constant from (i) and D is given by property (iii) of the boundary term.

Proof. Firstly note that as \mathcal{A} is a finite set and F is \mathcal{C} -invariant, the maximum $m := \max_{x \in G} \|F(\{x\})\|$ exists. Therefore we have for arbitrary $Q \in \mathcal{F}(G)$

$$\|F(Q)\| \leq \left\| F(Q) - \sum_{x \in Q} F(\{x\}) \right\| + \left\| \sum_{x \in Q} F(\{x\}) \right\| \leq \sum_{x \in Q} (b(\{x\}) + \|F(\{x\})\|) \leq C|Q|,$$

where $C := b(\{e\}) + m$ and e is the unit element in G . By definition of \tilde{F} this proves the second estimate in (i) as well. Now let $\varepsilon \in (0, 1/2)$ and ε -disjoint sets Q_i , $i = 1, \dots, k$ be given and set $Q = \bigcup_{i=1}^k Q_i$. Using the triangle inequality, we compute

$$\left\| F(Q) - \sum_{i=1}^k F(Q_i) \right\| \leq \left\| F(Q) - F\left(\bigcup_{i=1}^k \overline{Q}_i\right) \right\| + \left\| F\left(\bigcup_{i=1}^k \overline{Q}_i\right) - \sum_{i=1}^k F(\overline{Q}_i) \right\|$$

$$+ \sum_{i=1}^k \|F(Q_i) - F(\overline{Q}_i)\|. \quad (5.2)$$

Exploiting the fact that F is almost additive with boundary value b we obtain

$$\left\| F\left(\bigcup_{i=1}^k \overline{Q}_i\right) - \sum_{i=1}^k F(\overline{Q}_i) \right\| \leq \sum_{i=1}^k b(\overline{Q}_i)$$

as well as

$$\begin{aligned} \|F(Q_i) - F(\overline{Q}_i)\| &\leq \|F(Q_i) - F(\overline{Q}_i) - F(Q_i \setminus \overline{Q}_i)\| + \|F(Q_i \setminus \overline{Q}_i)\| \\ &\leq b(\overline{Q}_i) + b(Q_i \setminus \overline{Q}_i) + C |Q_i \setminus \overline{Q}_i| \\ &\leq b(\overline{Q}_i) + \varepsilon(C + D) |Q_i|. \end{aligned}$$

In the same manner, we derive

$$\begin{aligned} \left\| F(Q) - F\left(\bigcup_{i=1}^k \overline{Q}_i\right) \right\| &\leq b\left(\bigcup_{i=1}^k \overline{Q}_i\right) + b\left(Q \setminus \bigcup_{i=1}^k \overline{Q}_i\right) + C \left|Q \setminus \bigcup_{i=1}^k \overline{Q}_i\right| \\ &\leq b\left(\bigcup_{i=1}^k \overline{Q}_i\right) + \varepsilon(C + D) |Q|, \end{aligned}$$

where we applied the fact $|Q \setminus \bigcup_{i=1}^k \overline{Q}_i| \leq \varepsilon |Q|$ which is due to ε -disjointness. Putting the last estimates together and using property (iv) of the boundary term b , inequality (5.2) leads to

$$\left\| F(Q) - \sum_{i=1}^k F(Q_i) \right\| \leq 3 \sum_{i=1}^k b(\overline{Q}_i) + \varepsilon(C + D) \left(|Q| + \sum_{i=1}^k |Q_i| \right).$$

If we furthermore apply

$$b(\overline{Q}_i) \leq b(Q_i) + b(Q_i \setminus \overline{Q}_i) \leq b(Q_i) + D |Q_i \setminus \overline{Q}_i| \leq b(Q_i) + \varepsilon D |Q_i|,$$

which holds for all $i = 1, \dots, k$ again by property (iv) of the boundary term, we obtain

$$\begin{aligned} \left\| F(Q) - \sum_{i=1}^k F(Q_i) \right\| &\leq 3 \sum_{i=1}^k b(Q_i) + 3\varepsilon D \sum_{i=1}^k |Q_i| + \varepsilon(C + D) \left(|Q| + \sum_{i=1}^k |Q_i| \right) \\ &\leq 3 \sum_{i=1}^k b(Q_i) + \varepsilon(C + 4D) \sum_{i=1}^k |Q_i| + \varepsilon(C + D) |Q|. \end{aligned}$$

Using $\sum_{i=1}^k |Q_i| \leq (1 - \varepsilon)^{-1} |Q| \leq 2|Q|$, we arrive at the desired estimate. ■

For the sake of clarity, we summarize our major assumptions.

Assumption 5.4. Denote by G a countable amenable group, let \mathcal{A} be a finite set and consider a map $\mathcal{C} : G \rightarrow \mathcal{A}$, which will be called a coloring. Also, we assume that we are given a Følner sequence $(U_j)_{j \in \mathbb{N}}$ in G along which the frequencies ν_P exist for all patterns $P \in \mathcal{P}$. As pointed out before, $(X, \|\cdot\|)$ stands for a Banach-space.

Theorem 5.5. Assume 5.4 and let the function $F : \mathcal{F}(G) \rightarrow X$ be almost additive and \mathcal{C} -invariant and let \tilde{F} be given as in (5.1). Let $(S_n)_{n \in \mathbb{N}}$ be a nested Følner sequence. Then the following statements hold.

(i) There exists an element $\bar{F} \in X$ such that

$$\lim_{j \rightarrow \infty} \left\| \bar{F} - \frac{F(U_j)}{|U_j|} \right\| = 0.$$

(ii) The element \bar{F} can be expressed as the limit

$$\bar{F} = \lim_{\varepsilon \searrow 0} \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \nu_P \frac{\tilde{F}(P)}{|T_i^\varepsilon|},$$

where for each $0 < \varepsilon < 1/10$, we set $N(\varepsilon) := \lceil \log(\varepsilon)/\log(1 - \varepsilon) \rceil$ and $\eta_i(\varepsilon) := \varepsilon(1 - \varepsilon)^{N(\varepsilon)-i}$ for $i = 1, \dots, N(\varepsilon)$ and where the finite sequence $(T_i^\varepsilon)_{i=1}^{N(\varepsilon)}$ is given as in Definition 4.1 with parameters $\beta = 2^{-N(\varepsilon)-1}\varepsilon$ and $\delta_0(\beta)$. Each T_i^ε is an element of the sequence (S_n) .

(iii) For every $0 < \varepsilon < 1/10$, there is some $j_0 := j_0(\varepsilon, \beta) \in \mathbb{N}$ such that for every $j \geq j_0$, the difference

$$\Delta(j, \varepsilon) := \left\| \frac{F(U_j)}{|U_j|} - \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \nu_P \frac{\tilde{F}(P)}{|T_i^\varepsilon|} \right\|$$

satisfies the estimate

$$\Delta(j, \varepsilon) \leq (13C + 33D)\varepsilon + 4 \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \frac{b(T_i^\varepsilon)}{|T_i^\varepsilon|} + C \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \left| \frac{\#P(\mathcal{C}_{|U_j|})}{|U_j|} - \nu_P \right|. \quad (5.3)$$

Proof. Throughout the proof, we let $0 < \varepsilon < 1/10$ be fixed. We first show the Estimate (5.3). Choose $j_0 = j_0(\varepsilon, \beta, T_i^\varepsilon) \in \mathbb{N}$ such that for each $j \geq j_0$ the set U_j is sufficiently invariant to apply Theorem 4.7. Therefore for each $j \geq j_0$ we find a finite family Λ_j^ε of ε -quasi tilings for the set $T = U_j$ satisfying the uniform special tiling property (USTP), cf. Definition 4.6. With no loss of generality, we assume that all the $T_i = T_i^\varepsilon$ are taken from a subsequence $\{S_{n_k}\}_{k=1}^\infty$ such that the expressions $b(S_{n_k})/|S_{n_k}|$ converge to zero monotonically as $k \rightarrow \infty$. More precisely, we make sure that $T_i^\varepsilon \in \{S_{n_l} \mid l \geq i\}$ for all $1 \leq i \leq N$. Then, for fixed $j \geq j_0$ we estimate

$$\begin{aligned} \Delta(j, \varepsilon) &\leq \left\| \frac{F(U_j)}{|U_j|} - \frac{1}{|\Lambda_j^\varepsilon|} \sum_{\lambda \in \Lambda_j^\varepsilon} \sum_{i=1}^{N(\varepsilon)} \sum_{c \in C_i^\lambda(U_j), T_i^\varepsilon c \subseteq U_j} \frac{F(T_i^\varepsilon c)}{|U_j|} \right\| \\ &\quad + \left\| \frac{1}{|\Lambda_j^\varepsilon|} \sum_{\lambda \in \Lambda_j^\varepsilon} \sum_{i=1}^{N(\varepsilon)} \sum_{c \in C_i^\lambda(U_j), T_i^\varepsilon c \subseteq U_j} \frac{F(T_i^\varepsilon c)}{|U_j|} - \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \frac{\#P(\mathcal{C}_{|U_j|})}{|U_j|} \frac{\tilde{F}(P)}{|T_i^\varepsilon|} \right\| \end{aligned}$$

$$+ \left\| \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \left(\frac{\#_P(\mathcal{C}_{|U_j})}{|U_j|} - \nu_P \right) \frac{\tilde{F}(P)}{|T_i^\varepsilon|} \right\|.$$

Again by the triangle inequality, we then obtain

$$\Delta(j, \varepsilon) \leq D_1(j, \varepsilon) + D_2(j, \varepsilon) + D_3(j, \varepsilon),$$

where

$$D_1(j, \varepsilon) := \frac{1}{|U_j| |\Lambda_j^\varepsilon|} \sum_{\lambda \in \Lambda_j^\varepsilon} \left\| F(U_j) - \sum_{i=1}^{N(\varepsilon)} \sum_{c \in C_i^\lambda(U_j), T_i^\varepsilon c \subseteq U_j} F(T_i^\varepsilon c) \right\|,$$

$$D_2(j, \varepsilon) := \frac{1}{|U_j|} \left\| \frac{1}{|\Lambda_j^\varepsilon|} \sum_{\lambda \in \Lambda_j^\varepsilon} \sum_{i=1}^{N(\varepsilon)} \sum_{\substack{c \in C_i^\lambda(U_j) \\ T_i^\varepsilon c \subseteq U_j}} F(T_i^\varepsilon c) - \sum_{i=1}^{N(\varepsilon)} \frac{\eta_i(\varepsilon)}{|T_i^\varepsilon|} \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \#_P(\mathcal{C}_{|U_j}) \tilde{F}(P) \right\|,$$

and

$$D_3(j, \varepsilon) := \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \left| \frac{\#_P(\mathcal{C}_{|U_j})}{|U_j|} - \nu_P \right| \frac{\|\tilde{F}(P)\|}{|T_i^\varepsilon|}.$$

With the boundedness of \tilde{F} , see Lemma 5.3, we arrive at

$$D_3(j, \varepsilon) \leq C \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \left| \frac{\#_P(\mathcal{C}_{|U_j})}{|U_j|} - \nu_P \right|. \quad (5.4)$$

In order to estimate $D_1(j, \varepsilon)$ we make use of the almost additivity of the function F and use part (ii) of Lemma 5.3. So For each $j \geq j_0$ and $\lambda \in \Lambda_j^\varepsilon$ we have

$$\left\| F(U_j) - \sum_{i=1}^{N(\varepsilon)} \sum_{\substack{c \in C_i^\lambda(U_j) \\ T_i^\varepsilon c \subseteq U_j}} F(T_i^\varepsilon c) \right\| \leq \|F(U_j) - F(A_{j,\lambda}^\varepsilon)\| + \left\| F(A_{j,\lambda}^\varepsilon) - \sum_{i=1}^{N(\varepsilon)} \sum_{\substack{c \in C_i^\lambda(U_j) \\ T_i^\varepsilon c \subseteq U_j}} F(T_i^\varepsilon c) \right\|$$

$$\leq b(A_{j,\lambda}^\varepsilon) + b(U_j \setminus A_{j,\lambda}^\varepsilon) + \|F(U_j \setminus A_{j,\lambda}^\varepsilon)\| + 3 \sum_{i=1}^{N(\varepsilon)} \sum_{\substack{c \in C_i^\lambda(U_j) \\ T_i^\varepsilon c \subseteq U_j}} b(T_i^\varepsilon c) + (3C + 9D)\varepsilon|U_j|,$$

where

$$A_{j,\lambda}^\varepsilon = \bigcup_{i=1}^{N(\varepsilon)} \bigcup_{\substack{c \in C_i^\lambda(U_j) \\ T_i^\varepsilon c \subseteq U_j}} T_i^\varepsilon c.$$

Recall that by the USTP, for each $\lambda \in \Lambda$, the set U_j is $(1 - 4\varepsilon)$ -covered by those translates $T_i c$, $1 \leq i \leq N$, $c \in C_i^\lambda(U_j)$ that are fully contained in U_j . Therefore we have $|U_j \setminus A_{j,\lambda}^\varepsilon| \leq 4\varepsilon|U_j|$. Using this and properties of the boundary term b we obtain

$$\begin{aligned} D_1(j, \varepsilon) &\leq \frac{1}{|U_j||\Lambda_j^\varepsilon|} \sum_{\lambda \in \Lambda_j^\varepsilon} \left(4 \sum_{i=1}^{N(\varepsilon)} \sum_{\substack{c \in C_i^\lambda(U_j) \\ T_i^\varepsilon c \subseteq U_j}} b(T_i^\varepsilon c) + (7C + 13D)\varepsilon|U_j| \right) \\ &\leq \frac{4}{|U_j||\Lambda_j^\varepsilon|} \sum_{\lambda \in \Lambda_j^\varepsilon} \left(\sum_{i=1}^{N(\varepsilon)} |C_i^\lambda(U_j)| b(T_i^\varepsilon) \right) + (7C + 13D)\varepsilon. \end{aligned}$$

Summing up over U_j the expression in Definition 4.7, property (II), yields with $\beta = 2^{-N(\varepsilon)-1}\varepsilon$ that for each $j \geq j_0$, there are non-negative numbers $\gamma_i^{\varepsilon,j}$, $i = 1, \dots, N(\varepsilon)$, satisfying $\sum_{i=1}^{N(\varepsilon)} \gamma_i^{\varepsilon,j} |T_i^\varepsilon| \leq 2$ such that

$$\begin{aligned} D_1(j, \varepsilon) &\leq 4 \sum_{i=1}^{N(\varepsilon)} \frac{b(T_i^\varepsilon)}{|\Lambda_j^\varepsilon|} \sum_{\lambda \in \Lambda_j^\varepsilon} \frac{|C_i^\lambda(U_j)|}{|U_j|} + (7C + 13D)\varepsilon \\ &\leq 4 \sum_{i=1}^{N(\varepsilon)} b(T_i^\varepsilon) \left(\frac{\eta_i(\varepsilon)}{|T_i^\varepsilon|} + \frac{3\beta}{|T_i^\varepsilon|} + \varepsilon \gamma_i^{\varepsilon,j} \right) + (7C + 13D)\varepsilon \\ &\leq 4 \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \frac{b(T_i^\varepsilon)}{|T_i^\varepsilon|} + 12\beta D N(\varepsilon) + 8\varepsilon D + (7C + 13D)\varepsilon. \end{aligned}$$

By the choice of β we have $\beta N(\varepsilon) \leq \varepsilon$ and we arrive at

$$D_1(j, \varepsilon) \leq 4 \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \frac{b(T_i^\varepsilon)}{|T_i^\varepsilon|} + (7C + 33D)\varepsilon. \quad (5.5)$$

By exploiting the nice additional property (II) of Definition 4.6, we finally estimate $D_2(j, \varepsilon)$. With $C_i^\lambda(U_j) \subseteq U_j$ for $1 \leq i \leq N$ and $\lambda \in \Lambda$ (property (i)), we calculate

$$\begin{aligned} D_2(j, \varepsilon) &= \frac{1}{|U_j|} \left\| \sum_{i=1}^{N(\varepsilon)} \frac{1}{|\Lambda_j^\varepsilon|} \sum_{\lambda \in \Lambda_j^\varepsilon} \sum_{\substack{u \in U_j \\ T_i^\varepsilon u \subseteq U_j}} \mathbf{1}_{C_i^\lambda(U_j)}(u) F(T_i^\varepsilon u) - \sum_{i=1}^{N(\varepsilon)} \frac{\eta_i(\varepsilon)}{|T_i^\varepsilon|} \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \#_P(\mathcal{C}_{|U_j}) \tilde{F}(P) \right\| \\ &= \frac{1}{|U_j|} \left\| \sum_{i=1}^{N(\varepsilon)} \sum_{\substack{u \in U_j \\ T_i^\varepsilon u \subseteq U_j}} \frac{1}{|\Lambda_j^\varepsilon|} \sum_{\lambda \in \Lambda_j^\varepsilon} \mathbf{1}_{C_i^\lambda(U_j)}(u) F(T_i^\varepsilon u) - \sum_{i=1}^{N(\varepsilon)} \frac{\eta_i(\varepsilon)}{|T_i^\varepsilon|} \sum_{\substack{u \in U_j \\ T_i^\varepsilon u \subseteq U_j}} F(T_i^\varepsilon u) \right\| \\ &= \frac{1}{|U_j|} \left\| \sum_{i=1}^{N(\varepsilon)} \sum_{\substack{u \in U_j \\ T_i^\varepsilon u \subseteq U_j}} \left(\frac{1}{|\Lambda_j^\varepsilon|} \sum_{\lambda \in \Lambda_j^\varepsilon} \mathbf{1}_{C_i^\lambda(U_j)}(u) - \frac{\eta_i(\varepsilon)}{|T_i^\varepsilon|} \right) F(T_i^\varepsilon u) \right\|. \end{aligned}$$

We use the triangle inequality, the boundedness of F and afterwards the uniformity principle, property (II) of Definition 4.6 to obtain

$$\begin{aligned} D_2(j, \varepsilon) &\leq \frac{C}{|U_j|} \sum_{i=1}^{N(\varepsilon)} \sum_{\substack{u \in U_j \\ T_i^\varepsilon u \subseteq U_j}} \left| \frac{1}{|\Lambda_j^\varepsilon|} \sum_{\lambda \in \Lambda_j^\varepsilon} \mathbf{1}_{C_i^\lambda(U_j)}(u) - \frac{\eta_i(\varepsilon)}{|T_i^\varepsilon|} \right| \cdot |T_i^\varepsilon| \\ &\leq \frac{C}{|U_j|} \sum_{i=1}^{N(\varepsilon)} \sum_{\substack{u \in U_j \\ T_i^\varepsilon u \subseteq U_j}} \left(\frac{3\beta}{|T_i^\varepsilon|} + \varepsilon \gamma_i^{\varepsilon, j} \right) \cdot |T_i^\varepsilon|, \end{aligned}$$

such that with $\beta N(\varepsilon) \leq \varepsilon$ and $\sum_{i=1}^{N(\varepsilon)} \gamma_i^{\varepsilon, j} |T_i^\varepsilon| \leq 2$, we arrive at

$$D_2(j, \varepsilon) \leq 3C \cdot N(\varepsilon) \cdot \beta + \varepsilon C \sum_{i=1}^{N(\varepsilon)} \gamma_i^{\varepsilon, j} |T_i^\varepsilon| \leq 5\varepsilon C. \quad (5.6)$$

To finish the proof of (iii), we combine the inequalities (5.5), (5.6) and (5.4) and finally arrive at

$$\begin{aligned} \Delta(j, \varepsilon) &\leq D_1(j, \varepsilon) + D_2(j, \varepsilon) + D_3(j, \varepsilon) \\ &\leq (13C + 33D)\varepsilon + 4 \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \frac{b(T_i^\varepsilon)}{|T_i^\varepsilon|} + C \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \left| \frac{\#_P(\mathcal{C}_{|U_j|})}{|U_j|} - \nu_P \right| \end{aligned}$$

for all $j \geq j_0(\varepsilon, \beta, T_i^\varepsilon)$. Since $0 < \varepsilon < 1/10$ (and therefore also β) was arbitrarily chosen, this shows the desired estimate (5.3) for $\Delta(j, \varepsilon)$, $j \geq j_0(\varepsilon, \beta, T_i^\varepsilon)$.

Furthermore, Lemma 2.8 yields with the choice of the T_i^ε , as well as with the monotonicity assumption on the sequence $b(S_{n_k})/|S_{n_k}|$ (see above) that

$$\lim_{\varepsilon \searrow 0} \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) |T_i^\varepsilon|^{-1} b(T_i^\varepsilon) \leq \lim_{\varepsilon \rightarrow 0} \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) |S_{n_i}|^{-1} b(S_{n_i}) = 0.$$

This and the fact that the frequencies ν_P along $(U_j)_j$ exist shows with (5.3) that

$$\lim_{\varepsilon \rightarrow 0} \lim_{j \rightarrow \infty} \Delta(j, \varepsilon) = 0. \quad (5.7)$$

Now the triangle inequality shows that

$$\left\| \frac{F(U_j)}{|U_j|} - \frac{F(U_m)}{|U_m|} \right\| \leq \Delta(j, \varepsilon) + \Delta(m, \varepsilon)$$

for all $0 < \varepsilon < 1/10$ and every $j \geq j_0(\varepsilon)$. By (5.7) $(|U_j|^{-1} F(U_j))_{j \in \mathbb{N}}$ must be a Cauchy sequence and hence converges in the Banach space X to some element \bar{F} . The limit in (5.7) also shows that the expression

$$\sum_{i=1}^{N(\varepsilon)} \varepsilon (1 - \varepsilon)^{N(\varepsilon) - i} \sum_{P \in \mathcal{P}(T_i)} \nu_P \frac{\tilde{F}(P)}{|T_i|}$$

must converge in X to the same \bar{F} as $\varepsilon \rightarrow 0$. ■

We can use Inequality (5.3) to obtain explicit bounds on the speed of convergence. This will be shown in the next corollary.

Corollary 5.6. *In the situation of Theorem 5.5, the following estimates hold true:*

$$\left\| \bar{F} - \frac{F(U_j)}{|U_j|} \right\| \leq (26C + 66D)\varepsilon + 8 \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \frac{b(T_i^\varepsilon)}{|T_i^\varepsilon|} + C \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \left| \frac{\#P(\mathcal{C}_{|U_j|})}{|U_j|} - \nu_P \right|$$

for $j \geq j_0(\varepsilon)$ and

$$\left\| \bar{F} - \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \nu_P \frac{\tilde{F}(P)}{|T_i^\varepsilon|} \right\| \leq (13C + 33D)\varepsilon + 4 \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \frac{b(T_i^\varepsilon)}{|T_i^\varepsilon|}.$$

Proof. Fix $0 \leq \varepsilon < 1/10$. For the first estimate we get, using the definition of $\Delta(\cdot, \cdot)$, as well as the triangle inequality

$$\left\| \bar{F} - \frac{F(U_j)}{|U_j|} \right\| = \lim_{k \rightarrow \infty} \left\| \frac{F(U_k)}{|U_k|} - \frac{F(U_j)}{|U_j|} \right\| \leq \lim_{k \rightarrow \infty} [\Delta(k, \varepsilon) + \Delta(j, \varepsilon)].$$

Now the Estimate (5.3) yields the desired bound. To verify the second bound we write

$$\left\| \bar{F} - \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \nu_P \frac{\tilde{F}(P)}{|T_i^\varepsilon|} \right\| = \lim_{k \rightarrow \infty} \left\| \frac{F(U_k)}{|U_k|} - \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \nu_P \frac{\tilde{F}(P)}{|T_i^\varepsilon|} \right\| = \lim_{k \rightarrow \infty} \Delta(k, \varepsilon)$$

and again, the claim follows immediately from (5.3). ■

6 Sufficient conditions for the existence of frequencies

In this section we use the Lindenstrauss pointwise ergodic theorem, cf. [Lin01] to prove the existence of frequencies in a randomly coloured graph along a tempered Følner sequence. This is motivated by the Banach space-valued ergodic theorem in the previous section, as the existence of the frequencies are a basic assumption for its validity.

We consider a countable, amenable group G , as well as a probability space $(\Omega, \mathcal{S}, \mu)$. Let $\tau : G \times \Omega \rightarrow \Omega$, $(g, \omega) \mapsto \tau_g(\omega)$ be an action of G on Ω . We say that τ is *measure preserving*, if for any $A \in \mathcal{S}$ and $g \in G$ one has $\mu(A) = \mu(\tau_g(A))$. Furthermore, a measure preserving action is said to be *ergodic* if $\mu(A) \in \{0, 1\}$, whenever $A \in \mathcal{S}$ with $A = \tau_g^{-1}(A)$ for all $g \in G$.

In this situation, the Lindenstrauss ergodic theorem then reads as follows.

Theorem 6.1 (Lindenstrauss' pointwise ergodic theorem). *Let G be a countable amenable group and let τ be a measure preserving and ergodic action of G on the probability space $(\Omega, \mathcal{S}, \mu)$. Furthermore let (Q_j) be a tempered Følner sequence and $f \in L^1(\mu)$. Then one has for μ -almost all ω*

$$\lim_{j \rightarrow \infty} \frac{1}{|Q_j|} \sum_{g \in Q_j} f(\tau_g \omega) = \int_{\Omega} f(\omega) d\mu(\omega).$$

All proofs of the above theorem are based on the verification of a so-called maximal inequality via abstract, geometric covering arguments for very invariant subsets of the group, see e.g. [Lin01] and [Wei03]. Hence, one might wonder if one can use the ε -quasi tiling arguments of Section 4 to give a direct proof for a pointwise ergodic theorem. However, as one can infer from the literature, the assumptions on the corresponding decompositions can be kept much milder than in the situation of Definition 4.1 or even of Definition 4.6. As a consequence, by giving effective estimates for the ε -quasi tilings, we have to pay the price of strong invariance conditions. More precisely, for decreasing ε , the tiles used in this ε -quasi tiling are taken from a Følner (sub-)sequence consisting of increasingly invariant sets, such that the notion of temperedness is too mild to describe this condition. In that sense, our decomposition results should be considered as complementary to the classical results: we obtain more information on the shape, as well as on the degree of uniformity of the ε -quasi tilings, but we lose the properties of a tempered Følner sequence which lead to a maximal inequality.

In the following, we use the above ergodic theorem to show that frequencies of patterns exist almost surely in an appropriate probability space. We consider a countable amenable group G and a finite set \mathcal{A} , which we will interpret as the set of colours. The probability space $(\Omega, \mathcal{S}, \mathbb{P})$ is given in the following way. The sample space is the set

$$\Omega = \mathcal{A}^G = \{\omega = (\omega_g)_{g \in G} \mid \omega_g \in \mathcal{A} \text{ for all } g \in G\}.$$

The sigma-algebra \mathcal{S} is generated by the cylinder sets and \mathbb{P} is a probability measure on (Ω, \mathcal{S}) . Setting for each $\omega \in \Omega$

$$\mathcal{C}_\omega : G \rightarrow \mathcal{A}, \quad g \mapsto \omega_g,$$

shows that each ω can be interpreted as a colouring of G . Let $\tau : G \times \Omega \rightarrow \Omega$ be given by

$$(g, \omega) \mapsto \tau_g \omega = \omega g^{-1}, \tag{6.1}$$

where $\omega g^{-1} \in \Omega$ is the element satisfying

$$(\omega g^{-1})_x = \omega_{xg} \quad (x \in G).$$

We assume that the action τ of G on Ω is measure preserving and ergodic.

Using Theorem 6.1 we can prove the existence of the frequencies ν_P along any tempered Følner sequence (Q_j) . This has been shown in similar situations for example in [LSV10, Sch12, PSS11].

Theorem 6.2. *Let the probability space $(\Omega, \mathcal{S}, \mathbb{P})$ be given and let the action τ of G on Ω be measure preserving and ergodic. Furthermore let (Q_j) be a tempered Følner sequence. Then there exists a set $\tilde{\Omega}$ of full measure such that the limit*

$$\lim_{n \rightarrow \infty} \frac{\#_P \left((\mathcal{C}_\omega)|_{Q_j} \right)}{|Q_j|}$$

exists for all $P \in \mathcal{P}$ and all $\omega \in \Omega$ and the limit is independent of the specific choice of ω .

Proof. Let $P : D(P) \rightarrow \mathcal{A}$ be some pattern. As the number of occurrences of two equivalent patterns P_1 and P_2 in another pattern P_3 is the same, we can assume without loss of generality

that $\text{id} \in D(P)$. Set $A_P := \{\omega \in \Omega \mid (\mathcal{C}_\omega)|_{D(P)} = P\}$ and let $f_P : \Omega \rightarrow \{0, 1\}$ be the indicator function of A_P . Now we can estimate the number of occurrences of P in $(\mathcal{C}_\omega)|_{Q_j}$ by

$$\sum_{g \in Q_j} f_P(\omega g^{-1}) - |\partial_{D(P)} Q_j| \leq \sum_{g \in Q_j \setminus (\partial_{D(P)} Q_j)} f_P(\omega g^{-1}) \leq \sharp_P \left((\mathcal{C}_\omega)|_{Q_j} \right) \leq \sum_{g \in Q_j} f_P(\omega g^{-1}) \quad (6.2)$$

We apply Theorem 6.1, which is possible as τ acts measure preservingly and ergodically and since $f_P \in L^1(\mathbb{P})$. This yields that there is a set Ω_P of full measure such that

$$\lim_{j \rightarrow \infty} \frac{1}{|Q_j|} \sum_{g \in Q_j} f_P(\omega g^{-1}) = \mathbb{E}(f_P)$$

holds for all $\omega \in \Omega_P$. Using this with (6.2) and the fact that (Q_j) is a Følner sequence we obtain

$$\lim_{j \rightarrow \infty} \frac{\sharp_P \left((\mathcal{C}_\omega)|_{Q_j} \right)}{|Q_j|} = \mathbb{E}(f_P)$$

for all $\omega \in \Omega_P$. Next, set $\tilde{\Omega} = \bigcup_{P \in \mathcal{P}} \Omega_P$ and use the fact that \mathcal{P} is countable to get the desired set $\tilde{\Omega}$ of full measure such that the frequencies along (Q_j) exist for all patterns $P \in \mathcal{P}$ and all $\omega \in \tilde{\Omega}$. The independence of the specific choice of ω is clear as $\mathbb{E}(f_P)$ is independent of ω . ■

Remark 6.3. In the case where the measure \mathbb{P} has a product structure $\mathbb{P} = \prod_{g \in G} \mu$ and μ is some measure on \mathcal{A} , it is easy to show that τ , defined as in (6.1) is measure preserving and ergodic. This shows that Theorem 6.2 applies in particular to i.i.d. models.

7 Integrated density of states

In this section we are interested in the approximation of the integrated density of states via finite volume analogs. We show that for each finitely generated amenable group the approximants converge uniformly in the energy variable to a certain limit function. The topic has been investigated by LENZ, SCHWARZENBERGER, VESELIĆ in their work [LSV10] in a far more restricted framework. In the following, we will show briefly how the Ergodic Theorem 5.5 implies uniform convergence of the approximating functions. For further details, we refer to [LSV10]. We will also stick close to the notation in [LSV10].

Let \mathcal{H} be a finite dimensional Hilbert space with norm $\|\cdot\|$ and denote by $\ell^2(G, \mathcal{H})$ the Hilbert space of the functions $u : G \rightarrow \mathcal{H}$ such that $\sum_{g \in G} \|u(g)\|^2 < \infty$ for the usual ℓ^2 -norm. For given $Q \in \mathcal{F}(G)$ and an operator $H : \ell^2(G, \mathcal{H}) \rightarrow \ell^2(G, \mathcal{H})$ set

$$H[Q] := p_Q H i_Q : \ell^2(Q, \mathcal{H}) \rightarrow \ell^2(Q, \mathcal{H}),$$

where $i_Q : \ell^2(Q, \mathcal{H}) \rightarrow \ell^2(G, \mathcal{H})$ and $p_Q : \ell^2(G, \mathcal{H}) \rightarrow \ell^2(Q, \mathcal{H})$ are the canonical inclusions and projections given by

$$i_Q(v)(x) := \begin{cases} v(x) & \text{if } x \in Q \\ 0 & \text{else} \end{cases} \quad p_Q(u)(x) := u(x) \quad \text{for all } x \in Q$$

and $\ell^2(Q, \mathcal{H})$ is the space of all functions $v : Q \rightarrow \mathcal{H}$. Furthermore we set $H(x, y) := p_{\{x\}} H i_{\{y\}}$ for all $x, y \in G$.

Definition 7.1. Let G be a finitely generated amenable group, $d : G \times G \rightarrow \mathbb{N}_0$ the induced word metric, \mathcal{A} some finite set, $\mathcal{C} : G \rightarrow \mathcal{A}$ a coloring and assume that $H : \ell^2(G, \mathcal{H}) \rightarrow \ell^2(G, \mathcal{H})$ is self-adjoint. Then we say H is of *finite range* if there is some $M \in \mathbb{N}$ such that $d(g, h) \geq M$ implies $H(g, h) = 0$. Furthermore H is called *\mathcal{C} -invariant* if there is some $N \in \mathbb{N}$ such that $H(g, h) = H(gx, hx)$ holds whenever

$$(\mathcal{C}|_{B_N(g) \cup B_N(h)})x = \mathcal{C}|_{B_N(gx) \cup B_N(hx)}.$$

If both conditions are fulfilled, then we call $R = \max\{M, N\}$ the *overall range* of H .

Definition 7.2. Let $\mathcal{B}(\mathbb{R})$ denote the Banach space of right-continuous, bounded functions $f : \mathbb{R} \rightarrow \mathbb{R}$, equipped with the supremum norm. Given a self-adjoint operator A on a finite dimensional Hilbert space V , we define its cumulative eigenvalue counting function $n(A) \in \mathcal{B}(\mathbb{R})$ by setting

$$n(A)(E) := |\{i \in \mathbb{N} \mid \lambda_i \leq E\}|$$

for all $E \in \mathbb{R}$, where $\lambda_i, i = 1, \dots, \dim V$ are the eigenvalues of A , counted according to their multiplicity.

Assumption 7.3. Assume 5.4 and additionally that G is finitely generated, \mathcal{H} is a finite dimensional Hilbert space, $H : \ell^2(G, \mathcal{H}) \rightarrow \ell^2(G, \mathcal{H})$ is a self-adjoint, finite range and \mathcal{C} -invariant operator with overall range R .

Proposition 7.4. Assume 7.3. The function $F_R^H : \mathcal{F}(G) \rightarrow \mathcal{B}(\mathbb{R})$, $Q \mapsto F_R^H(Q) := n(H[Q_R])$ is \mathcal{C} -invariant and almost-additive with the boundary term $b(Q) := 4|\partial^R Q| \dim(\mathcal{H})$.

Proof. see [LSV10], Proposition 4.6. ■

Note that indeed, this $b : \mathcal{F}(G) \rightarrow [0, \infty)$ is a boundary term since $\lim_{j \rightarrow \infty} |\partial^R(Q_j)|/|Q_j| = 0$ for any Følner sequence (Q_j) , c.f. [LSV10]. Furthermore b is obviously invariant under translations and we have $b(Q) \leq D|Q|$ with $D = 4 \dim(\mathcal{H})|B_R|$. Also property (iv) of Definition 5.1 is true for b as it holds for the function $|\partial^R(\cdot)|$. Moreover, we know from the definition of F_R^H that $F_R^H(Q) \leq C|Q|$, where $C := \dim(\mathcal{H})$.

Theorem 7.5. Assume 7.3. Then there exists a unique probability measure μ_H with distribution function N_H , such that

$$\lim_{j \rightarrow \infty} \left\| \frac{n(H[U_{j,R}])}{\dim(\mathcal{H})|U_j|} - N_H \right\|_\infty = 0$$

The function N_H is called integrated density of states. Furthermore for each $0 < \varepsilon < 1/10$ there is a constant $j_0 = j_0(\varepsilon)$ such that the estimate

$$\left\| \frac{n(H[U_{j,R}])}{\dim(\mathcal{H})|U_j|} - N_H \right\|_\infty \leq (26 + 264|B_R|)\varepsilon + \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \left| \frac{\#P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \right| + \sum_{i=1}^{N(\varepsilon)} \frac{8\eta_i(\varepsilon)b(T_i^\varepsilon)}{\dim(\mathcal{H})|T_i^\varepsilon|}$$

holds for all $j \geq j_0$. As before we have $N(\varepsilon) := \lceil \log(\varepsilon)/\log(1-\varepsilon) \rceil$ and $\eta_i(\varepsilon) := \varepsilon(1-\varepsilon)^{N(\varepsilon)-i}$ for $i = 1, \dots, N(\varepsilon)$. The sets $T_i^\varepsilon, i = 1, \dots, N(\varepsilon)$ are given as in Definition 4.1 with parameters $\beta = 2^{-N(\varepsilon)-1}\varepsilon$ and $\delta_0(\beta)$ and a nested Følner sequence (S_n) .

Proof. By Proposition 7.4 we have that $F_R^H : \mathcal{F}(G) \rightarrow \mathcal{B}(\mathbb{R})$, $Q \mapsto F_R^H(Q) := n(H[Q_R])$ is \mathcal{C} -invariant and almost-additive with the boundary term $b(Q) := 4|\partial^R Q| \dim(\mathcal{H})$. Hence we can apply Theorem 5.5 and Corollary 5.6 in order to find a function $\tilde{N}_H \in \mathcal{B}(\mathbb{R})$ such that

$$\begin{aligned} \left\| \tilde{N}_H - \frac{F_R^H(U_j)}{|U_j|} \right\|_\infty &\leq (26C + 66D)\varepsilon + 8 \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \frac{b(T_i^\varepsilon)}{|T_i^\varepsilon|} + C \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \left| \frac{\#_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \right| \\ &\leq \dim(\mathcal{H}) \left((26 + 264|B_R|)\varepsilon + \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \left| \frac{\#_P(\mathcal{C}|_{U_j})}{|U_j|} - \nu_P \right| \right) + 8 \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \frac{b(T_i^\varepsilon)}{|T_i^\varepsilon|} \end{aligned}$$

This proves the claimed estimate with $N_H = \frac{\tilde{N}_H}{\dim(\mathcal{H})}$. As in the proof of Theorem 5.5, we may assume without loss of generality that $|S_n|^{-1}b(S_n)$ converges monotonically to zero, which gives by the choice of the T_i^ε and by Lemma 2.8 that

$$\lim_{\varepsilon \searrow 0} \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \frac{b(T_i^\varepsilon)}{|T_i^\varepsilon|} = 0,$$

This shows

$$\lim_{j \rightarrow \infty} \left\| N_H - \frac{F_R^H(U_j)}{\dim(\mathcal{H})|U_j|} \right\|_\infty = 0.$$

It remains to prove that N_H is the distribution function of a probability measure. The monotonicity of N_H is clear since we have for any $E' \leq E$

$$N_H(E) - N_H(E') = \lim_{j \rightarrow \infty} \frac{F_R^H(U_j)(E) - F_R^H(U_j)(E')}{\dim(\mathcal{H})|U_j|} \geq 0,$$

as the functions $F_R^H(U_j)$ are monotone. By the uniform convergence, the right continuity of the functions $F_R^H(U_j)$ carries over to the limit N_H . In fact we use the uniform convergence to interchange the limits in the computation:

$$\lim_{E' \searrow E} N_H(E') = \lim_{E' \searrow E} \lim_{j \rightarrow \infty} \frac{F_R^H(U_j)(E')}{\dim(\mathcal{H})|U_j|} = \lim_{j \rightarrow \infty} \lim_{E' \searrow E} \frac{F_R^H(U_j)(E')}{\dim(\mathcal{H})|U_j|} = N_H(E).$$

Another application of uniform convergence yields

$$\lim_{E \rightarrow \infty} N_H(E) = \lim_{E \rightarrow \infty} \lim_{j \rightarrow \infty} \frac{F_R^H(U_j)(E)}{\dim(\mathcal{H})|U_j|} = \lim_{j \rightarrow \infty} \lim_{E \rightarrow \infty} \frac{F_R^H(U_j)(E)}{\dim(\mathcal{H})|U_j|} = \lim_{j \rightarrow \infty} \frac{\dim(\mathcal{H})|U_{j,R}|}{\dim(\mathcal{H})|U_j|} = 1.$$

Similarly, one obtains $\lim_{E \rightarrow -\infty} N_H(E) = 0$. Hence, N_H is the distribution function of a probability measure. \blacksquare

For completeness reasons let us state two other results, concerning properties of the integrated density of states. Both are taken from [LSV10], where one also finds the proofs, which still hold true in the present setting.

Assumption 7.6. Assume 7.3 and additionally that the frequencies ν_P are strictly positive for all patterns $P \in \mathcal{P}$ which occur in \mathcal{C} , i.e. for which there exists $g \in G$ with $\mathcal{C}|_{D(P)g} = Pg$.

Theorem 7.7. If we assume 7.6, then the spectrum of H equals the topological support of μ_H .

Corollary 7.8. Assume 7.6 and let $\lambda \in \mathbb{R}$. Then E is a point of discontinuity of N_H , if and only if there exists a compactly supported eigenfunction of H corresponding to E .

8 A periodic model

In this section, we use Theorem 5.5 to prove the Banach space valued approximation of the integrated density of states (IDS) for finite hopping range operators on discrete structures with a quasi isometry to an amenable group. The model is taken from [LV09], where the authors consider randomly chosen discrete sets $X(\omega)$ in a general metric space (X, d_X) being quasi isometric to some unimodular amenable group G , endowed with a metric d_G . More precisely, X and G are related by some relatively compact fundamental domain F' . Restricting ourselves to discrete structures and avoiding randomness, we use the Banach space valued ergodic theorem to prove the uniform approximation of the IDS in $\mathcal{B}(\mathbb{R})$, the Banach space of right-continuous, bounded functions, endowed with the supremum norm, c.f. Theorem 8.6. Moreover, having the Banach space valued ergodic theorem at hand, there is no need to use a measure theoretical machinery as in [LV09] (Lemmas 6.1, 6.2 and 6.3) converting weak convergence into uniform convergence. However, such arguments are necessary to identify the limit as an expression depending only on the fundamental domain and on the operator under consideration, cf. Remark 8.7. As it is convenient to work with right fundamental domains for our purposes, we convey the linear algebra arguments of [LV09] to the corresponding general setting (cf. Subsection 8.1) before drawing our attention to discrete spaces only (cf. Subsection 8.2).

8.1 Quasi isometry and linear algebra

Definition 8.1. Let (X, d_X) be a locally compact, metric space with a countable basis of the topology and let G be a locally compact amenable unimodular group with an invariant metric d_G such that every ball is precompact. We then say that G acts continuously from the right by isometries on X if:

- there exists a right fundamental domain F' with compact closure F , which is a countable union of compact sets
- the map $\Phi : X \rightarrow G : x \mapsto g$, whenever $x \in F'g$, is a *quasi isometry*, i.e. there exist $a \geq 1$ and $b \geq 0$ with

$$\frac{1}{a} d_G(\Phi(x), \Phi(y)) - b \leq d_X(x, y) \leq a d_G(\Phi(x), \Phi(y)) + b$$

for all $x, y \in X$.

For a set $\Lambda \subseteq X$, $x \in X$ and $r > 0$, we define $d_X(x, \Lambda) = \inf\{d_X(x, y) \mid y \in \Lambda\}$ and

$$\Lambda^r := \{x \in X \mid d_X(x, \Lambda) < r\}, \Lambda_r := \{x \in X \mid d_X(x, X \setminus \Lambda) > r\}, \partial^r \Lambda := \Lambda^r \setminus \Lambda_r.$$

For $Q \subseteq G$ we use the analogous notations Q^r, Q_r and $\partial^r(Q)$ with d_X replaced by d_G . We denote by $B_s(g)$ the open ball around $g \in G$ with radius s . If $g = \text{id}$, we simply write B_s for $B_s(\text{id})$. Further, for $s > 0$ and $\eta > 0$, we denote by $M_s(\eta)$ the maximal number of points with distance at least $\eta > 0$ contained in a ball of radius s in X . Note that for all choices of parameters, this is a finite number.

The following proposition can be proven along the lines of [LV09].

Proposition 8.2. *Let (Q_n) be a Følner sequence in G . Then for arbitrary $\rho, r > 0$ and each uniformly discrete set $A \subseteq X$, i.e. $d_X(x, y) \geq \eta$ for all $x, y \in A$ with $x \neq y$,*

$$|A \cap F'(\partial^r Q_n)| \leq \frac{M_{a\rho+b}(\eta)}{|B_{\rho/2}|} |(\partial^r Q_n)^\rho| \leq \frac{M_{a\rho+b}(\eta)}{|B_{\rho/2}|} |(\partial^{r+\rho} Q_n)|$$

for all $n \in \mathbb{N}$. In particular, we have

$$\lim_{n \rightarrow \infty} \frac{|A \cap F'(\partial^r Q_n)|}{|Q_n|} = 0 \text{ and } \lim_{n \rightarrow \infty} \frac{|A \cap \partial^r(FQ_n)|}{|Q_n|} = 0.$$

Proof. See [LV09], Proposition 3.4. ■

Proposition 8.3 (cf. [LV09], Lemma 3.5). *Let $A \subseteq X$ be such that $d_X(x, y) \geq \eta$, whenever $x, y \in A$ with $x \neq y$. Then, if U is a subspace of $\ell^2(A \cap FQ)$ and if U_s is the subspace consisting of all functions in U which vanish outside of $(FQ)_s$ for $s \geq 0$, we obtain*

$$0 \leq \dim(U) - \dim(U_s) \leq |A \cap \partial^s(FQ)|.$$

Proof. Let $W : U \rightarrow \ell^2(A \cap ((FQ) \setminus (FQ)_s))$ be the natural restriction map. Then obviously

$$\dim(U) - \dim(\ker W) = \dim(\text{ran } W).$$

As $\ker(W) = U_s$ and $\dim(\text{ran } W) \leq |A \cap (FQ \setminus (FQ)_s)| \leq |A \cap \partial^s(FQ)|$, the statement follows. ■

8.2 Banach space approximation of the IDS

In the following, we treat a special case of the situation described in Definition 8.1. Namely, we take a graph $\Gamma = (V, E)$ with a countable vertex set V and edge set E . As above, we denote by G some finitely generated amenable group with symmetric generating system S and we assume that G acts freely and cocompactly on Γ by automorphisms. Further, G shall be endowed with the canonical word metric $d_G := d_S$. We set $X := V$ and show that the assumptions of our setting are satisfied.

Since G acts cocompactly, there is a finite right fundamental domain $F' = F$ of the action of G . As G acts freely, we obtain a uniquely defined map

$$\Phi : X \rightarrow G : x \mapsto g, \quad x \in Fg.$$

We assume first that Γ is connected and endow the vertex set X with the canonical path metric d_X determined by the edges. Using the fact that all word metrics on G arising from a finite generating set are equivalent (and therefore quasi-isometric), it is shown in [LV09] (Section 9) that in this case Φ is a quasi isometry for the metrics d_G on G and d_X on X .

If Γ is not connected, we induce a metric d_X on X as follows. The word metric d_G on the group defines a distance between two fundamental domains. Within the fundamental domain F we measure the distance between two points by the discrete metric d_F . In this way one obtains a metric

$$d_X(x, y) = d_G(\Phi(x), \Phi(y)) + d_F(x \cdot \Phi(x)^{-1}, y \cdot \Phi(y)^{-1})$$

on X which is by construction quasi isometric to d_G (cf. [LV09], Section 9). Moreover, this metric turns X into a uniformly discrete space with $\eta = 1$.

We assume further that we are given a family $s_g, g \in G$ of maps

$$s_g : X \rightarrow \{z \in \mathbb{C} \mid |z| = 1\}.$$

Moreover, let H be a self-adjoint operator on $\ell^2(X)$ which is of finite hopping range $R > 0$, i.e. $Hu(x) := \sum_{y \in X} h(x, y) u(y)$ for $u \in \ell^2(X)$ and $h(x, y) = 0$, whenever $d_X(x, y) \geq R$. We also assume that H possesses some kind of a shifting invariance, i.e. $T_g H = H T_g$ for all $g \in G$, where

$$T_g : \ell^2(X) \rightarrow \ell^2(X) : (T_g u)(x) := s_g(xg) u(xg).$$

For $R > 0$ and $Q \subseteq G$, we then define the operator $H[Q]$ as

$$H[Q] := p_{(FQ)_R} H i_{(FQ)_R}.$$

Further, we set

$$N_H^R(Q)(\lambda) := \text{tr}(\mathbf{1}_{]-\infty, \lambda]}(H[Q])) = |\{i \in \mathbb{N} \mid E_i \text{ e.v. of } H[Q], E_i \leq \lambda\}|$$

for $R > 0$, $Q \subseteq G$ and $\lambda \in \mathbb{R}$.

Theorem 8.4. *Let $0 < R < \infty$ be the finite hopping range of H . Then for each $\lambda \in \mathbb{R}$, the function $N_H^R(\cdot)(\lambda)$ is*

- (i) *invariant, i.e. $N_H^R(Qg)(\lambda) = N_H^R(Q)(\lambda)$ for all finite $Q \subseteq G$ and all $g \in G$.*
- (ii) *almost additive with boundary term $b(Q) = C|\partial^{\bar{R}}(Q)|$ for finite $Q \subseteq G$, where the constants $C, \bar{R} > 0$ depend on a, b and R only.*

Remark 8.5. It follows then from Lemma 5.3 (i) (Case $|\mathcal{A}| = 1$) that there is some constant $C > 0$ such that $\|N_H^R(Q)\|_\infty \leq C|Q|$ for all finite $Q \subseteq G$.

Proof. (i): Let $Q \subseteq G$ and $g \in G$. We set $B := FQ$ and we assume that $u \in \ell^2(B_R)$. Since G acts on Γ by isometries, we have $B_R h = (Bh)_R$ for every $h \in G$. Hence, it is enough to consider the operator $p_{Dg} H i_{Dg}$ with $D := B_R$ and to show that it is nothing but the operator $p_D H i_D$ shifted in its domain by g . To do so, take $\tilde{u} \in \ell^2(Dg)$ with $\tilde{u}(bg) := u(b)$ for $b \in D$ and compute for all $b \in D$

$$\begin{aligned} (p_{Dg} H i_{Dg}) \tilde{u}(bg) &= (H i_{Dg}) \tilde{u}(bg) \\ &= [s_g(bg)]^{-1} (T_g H i_{Dg}) \tilde{u}(b) \\ &\stackrel{\text{inv.}}{=} [s_g(bg)]^{-1} (H T_g i_{Dg}) \tilde{u}(b) \\ &= [s_g(bg)]^{-1} (H s_g(\cdot g)) i_D u(b) \\ &= (H i_D) u(b) = (p_D H i_D) u(b), \end{aligned}$$

which shows our claim.

(ii): We assume that $Q := \cup_{k=1}^m Q_k$ is a disjoint union of sets $Q_k \subseteq G$.

Hence, with Proposition 7.2 in [LSV10] with $V := l^2((FQ)_R)$ and $U := l^2(\cup_{k=1}^m (FQ_k)_R)$ and using Proposition 8.2, we compute

$$\begin{aligned} \left| \operatorname{tr}(\mathbf{1}_{]-\infty, \lambda]} H[Q]) - \sum_{k=1}^m \operatorname{tr}(\mathbf{1}_{]-\infty, \lambda]} H[Q_k]) \right| &= \left| \operatorname{tr}(\mathbf{1}_{]-\infty, \lambda]} H[Q]) - \operatorname{tr} \left(\bigoplus_{k=1}^m \mathbf{1}_{]-\infty, \lambda]} H[Q_k] \right) \right| \\ &\leq 4 \sum_{k=1}^m |\partial^R(FQ_k)| \stackrel{8.2}{\leq} C(a, b, R) \sum_{k=1}^m |\partial^{\bar{R}} Q_k|. \end{aligned}$$

for some $C, \bar{R} > 0$ chosen according to Proposition 8.2.

It is clear that the map $b : \mathcal{F}(G) \rightarrow [0, \infty)$ satisfies the properties (i), (ii) and (iv) of Definition 5.1. For the boundedness (iii), note that by discreteness of G , we have $\partial^{\bar{R}}(Q) \subseteq \partial_{B_{2\bar{R}}}(Q)$ and thus, $|\partial^{\bar{R}}(Q)| \leq D |Q|$ with $D := |B_{2\bar{R}}|$. \blacksquare

We now state our convergence result. To do so, denote by $\mathcal{B}(\mathbb{R})$ the Banach space of bounded, right continuous functions on the real line, endowed with the supremum norm $\|\cdot\|_\infty$.

Theorem 8.6. *If (Q_k) is a Følner sequence in G , the sequence $N_H^R(Q_k)(\lambda)/|FQ_k|$ converges in $\mathcal{B}(\mathbb{R})$ to some right continuous, bounded function $\tilde{N}(\cdot)$ on \mathbb{R} , i.e.*

$$\lim_{k \rightarrow \infty} \left\| \frac{N_H^R(Q_k)(\cdot)}{|FQ_k|} - \tilde{N} \right\|_\infty = \lim_{k \rightarrow \infty} \sup_{\lambda \in \mathbb{R}} \left| \frac{N_H^R(Q_k)(\lambda)}{|FQ_k|} - \tilde{N}(\lambda) \right| = 0.$$

Proof. Define the mapping

$$\hat{N}_H^R : \mathcal{F}(G) \rightarrow \mathcal{B}(\mathbb{R}) : Q \mapsto |F|^{-1} N_H^R(Q)(\cdot).$$

Since all $N_H^R(Q)/|Q|$ belong to $\mathcal{B}(\mathbb{R})$ for $Q \in \mathcal{F}(G)$, this map is well defined. By Theorem 8.4, it is also invariant and almost additive with the canonical boundary term. By the ergodic Theorem 5.5 with the trivial coloring (one color, $|\mathcal{A}| = 1$), the sequence $N_H^R(Q_k)/|FQ_k| = \hat{N}_H^R(Q_k)/|Q_k|$ converges in the $\|\cdot\|_\infty$ -norm to some $\tilde{N} \in \mathcal{B}(\mathbb{R})$ as k tends to infinity. \blacksquare

Remark 8.7. It follows from the considerations in [LV09] that the limit \tilde{N} can be expressed as

$$\tilde{N}(\lambda) := \operatorname{tr}(\mathbf{1}_F \mathbf{1}_{]-\infty, \lambda]} H)/|F|$$

for $\lambda \in \mathbb{R}$ (cf. [LV09], Theorem 2.4).

9 Approximation of ℓ^2 -Betti numbers

We use the convergence result, Theorem 8.6, of the previous section to show an approximation result for ℓ^2 -Betti numbers on cellular CW-complexes. This has been done before by different methods, see e.g. [DM98], [Eck99] and [Sch01].

Let X be a cellular CW-complex, i.e. a topological space consisting of finite dimensional cells (simplices).

Assume further that G is a discrete amenable group which acts freely and cocompactly from the right on X by permutations of the simplices. Hence the resulting simplicial complex $F := X/G$ is a finite fundamental domain. Denote by $X_n := F_n G$ the set of n -dimensional simplices of X , where the elements F_n are the n -cells in F . We call two such cells adjacent if

either the intersection of their closures contains a $(n - 1)$ -cell of X , or if both are contained in a closure of a single $(n + 1)$ -cell. In light of that, we can define a graph G_n with vertex set $V_n := X_n$, where two elements in V_n shall be connected by an edge if and only if they are adjacent. Note further that each automorphism on X induces a graph automorphism on G_n and G then acts freely and cocompactly on G_n . With the scaled path metric d on G_n as described in the previous section, it follows that the graph G_n is quasi-isometric to the group G .

For each $n \in \mathbb{N}$, we set $C_n(X)$ as the space of finitely supported complex-valued functions defined on elements in V_n . Moreover, we assume that the spaces $C_n(X)$, which can be seen as free $\mathbb{C}G$ -modules with basis elements in F_n , are connected by homomorphisms

$$\bar{\partial}_n : C_{n+1}(X) \rightarrow C_n(X)$$

with the property that $\bar{\partial}_n \circ \bar{\partial}_{n+1} = 0$ for all $n \in \mathbb{N}$. This defines a cellular chain complex $C^*(X)$ on X .

Following [DM98], we can find a natural inner product on $C_n(X)$, given by

$$\langle f, g \rangle := \sum_{\sigma \in V_n} f(\sigma) \overline{g(\sigma)}$$

for $f, g \in C_n(X)$. In a canonical manner, by taking the closure of this space w.r.t. the norm derived from this inner product, we obtain a Hilbert space $C_n^{(2)}(X) \cong \ell^2(V_n)$. From the boundary mappings of the chain complex $C^*(X)$, we can define the so-called *boundary operator*

$$\partial_n : C_{n+1}^{(2)}(X) \rightarrow C_n^{(2)}(X).$$

In this setting, the so-called *coboundary operator*

$$\partial_n^* : C_n^{(2)}(X) \rightarrow C_{n+1}^{(2)}(X)$$

is the formal adjoint of ∂_n . Then the (reduced) ℓ^2 -cohomology groups are defined as

$$H_n^{(2)}(X) := \frac{\ker(\partial_n^*)}{\text{Im}(\partial_{n-1}^*)}.$$

We define the combinatorial Laplacian $\Delta_n : C_n^{(2)}(X) \rightarrow C_n^{(2)}(X)$ as $\Delta_n := \partial_n \partial_n^* + \partial_{n-1}^* \partial_{n-1}$. This is a bounded, self-adjoint, positive, linear operator of finite hopping range $R = 2h$ (cf. [Sch01]), where $h > 0$ is the scale for the distance of two adjacent vertices in V_n . Further, Δ_n commutes with the operators T_g (for the definition, see the previous section).

In this context, we define the n -th ℓ^2 -Betti number $\bar{\beta}_n$ for the space X as

$$\bar{\beta}_n := \dim(\ker \Delta_n).$$

We now assume that $(Q_k)_k$ is a Følner sequence in G and set $X_k := FQ_k$ for $k \in \mathbb{N}$. It is shown in the following that for each $n \in \mathbb{N}$, we can find approximants $\beta_n(Q_k)$ such that

$$\lim_{k \rightarrow \infty} \frac{\beta_n(Q_k)}{|Q_k|} = \bar{\beta}_n.$$

With the notation in the previous section, we can do so by setting $\beta_n(Q_k) := N_{\Delta_n}^R(Q_k)(0)/|F_n|$ for $k \in \mathbb{N}$. This leads to the following theorem.

Theorem 9.1. *For each $n \in \mathbb{N}$, the limit*

$$\tilde{\beta}_n := \lim_{k \rightarrow \infty} \frac{\beta_n(Q_k)}{|Q_k|}$$

exists and $\tilde{\beta}_n = \overline{\beta}_n = \text{tr}(\mathbf{1}_{F_n} \mathbf{1}_{\{0\}} \Delta_n) / |F_n|$.

Proof. Let $n \in \mathbb{N}$. The existence of the limit $\tilde{\beta}_n$ follows from Theorem 8.6. The claimed representation $\tilde{\beta}_n = \overline{\beta}_n$ follows from Remark 8.7 and the fact that Δ_n is positive. ■

10 Distribution of percolation clusters

In this section we consider a bond percolation model for Cayley graphs given by finitely generated amenable groups. The aim is to study densities and distribution functions of percolation clusters. Parts of the results have been obtained for \mathbb{Z}^d (with site percolation) in the diploma thesis [Wei11] supervised by I. Veselić and F. Schwarzenberger. Our Theorem 5.5 allows us to generalize these ideas to arbitrary, finitely generated amenable groups. While we prove the existence of certain densities in the first subsection, the second subsection is devoted to a close look on the dependence of these densities on the percolation parameter. Note that in this section we treat finitely generated groups, such that the Haar measure $|\cdot|$ of a set equals the counting measure.

10.1 Distribution function and densities

Let G be an amenable group and suppose that $S \subseteq G$ is a finite and symmetric set of generators. As before, the set of all finite subsets of G is $\mathcal{F}(G)$. We denote the undirected graph by $\Gamma = \Gamma(G, S) = (V, E)$, where the vertex set V is equal to the set of elements in G and two vertices x and y are adjacent if there exists some $s \in S$ such that $sx = y$. We write $e = [x, y] = [y, x]$ for an edge connecting the vertices $x, y \in G$. Note that the induced graph metric on Γ corresponds to the word metric defined in Remark 2.3. In the following, we will also use the notation $\partial^r(\Lambda)$ for the r -boundary as introduced in 2.3.

To obtain a random subgraph we set

$$\Omega = \{0, 1\}^{V \times S} = \{(\omega_{v,s})_{v \in V, s \in S} \mid \omega_{v,s} \in \{0, 1\} \text{ for all } v \in V, s \in S\}$$

and consider the σ -algebra \mathcal{Z} which is generated by the cylinder sets. Furthermore, we set \mathbb{P} to be the product measure $\mathbb{P} = \prod_{v \in V, s \in S} \nu_s$, where ν_s is a Bernoulli measure with values $\{0, 1\}$. More precisely, we fix $(p_s)_{s \in S} \in [0, 1]^S$ and require ν_s to fulfill $\nu_s(X_{v,s} = 1) = p_s = 1 - \nu_s(X_{v,s} = 0)$ for all $v \in V$ and $s \in S$. Here for each pair $(v, s) \in V \times S$ the mapping $X_{v,s} : \Omega \rightarrow \{0, 1\}$ is the projection on the coordinate (v, s) .

In this situation each $\omega = (\omega_{v,s})_{v \in V, s \in S} \in \Omega$ defines a subgraph $\Gamma_\omega = (V, E_\omega)$ of Γ by setting $E_\omega = \{[x, y] \in E \mid X_{x, yx^{-1}} = X_{y, xy^{-1}} = 1\}$. Therefore we have for given $e = [x, sx] \in E$ with $s \in S \setminus \{\text{id}\}$ that $\mathbb{P}(e \in E_\omega) = p_s p_{s^{-1}}$ and (if $\text{id} \in S$) we have $\mathbb{P}([x, x] \in E_\omega) = p_{\text{id}}$. An edge $e \in E$ is called active in the configuration ω respectively in the graph Γ_ω if $e \in E_\omega$ and non-active otherwise. Note that in the case where there exists a $p \in [0, 1]$ such that $p_s^2 = p$ for all $s \in S \setminus \{\text{id}\}$ and (if $\text{id} \in S$) $p_{\text{id}} = p$, each edge $e \in E$ will be active with probability p .

Remark 10.1. Let us briefly explain the reasons for which we choose this probability space. First of all, our aim is to interpret each $\omega \in \Omega$ as a coloring of the graph with finitely many colors in order to apply the ergodic theorem developed in Section 5. Therefore, it is necessary to define Ω as a product over the set of vertices. Furthermore, we would like to allow for different probabilities determining the existence of the edges. However, the latter should be invariant under translation via multiplication of group elements. Hence it is useful to couple these probabilities with the generators. In the case where S only contains elements which are not self-inverse, this turns out to be easy. Then one could write S as the disjoint union of \bar{S} and \bar{S}^{-1} and express the set of all edges as $E = \bigcup_{v \in V, s \in \bar{S}} [v, sv]$. Hence it is enough to consider the space

$$\bar{\Omega} = \{0, 1\}^{V \times \bar{S}} = \{(\omega_{v,s})_{v \in V, s \in \bar{S}} \mid \omega_{v,s} \in \{0, 1\} \text{ for all } v \in V, s \in \bar{S}\},$$

along with the measure

$$\bar{\mathbb{P}} = \prod_{v \in V, s \in \bar{S}} \bar{\nu}_s \quad \text{where} \quad \bar{\nu}_s(X_{v,s} = 1) = p_s = 1 - \bar{\nu}_s(X_{v,s} = 0) \text{ and } p \in [0, 1]^{\bar{S}}.$$

Here again $X_{v,s}(\omega) = \omega_{v,s}$ for all $v \in V$ and $s \in \bar{S}$. In this case we would set an edge $[v, sv] \in E$ to be active in the configuration ω if and only if $X_{v,s}(\omega) = 1$. This strategy is not possible when S contains a self-inverse element as then one can not decompose S in some set \bar{S} and its inverse and hence the representation of E as a disjoint union over the set of vertices (as above) is not longer possible. In this case we can write $E = \bigcup_{v \in V, s \in S} [v, sv]$, which is a union that counts each edge (which is not a loop) twice. This leads to the probability space introduced above.

Given $e = [x, y] \in E$ and $z \in G$ we set $ez = [xz, yz] \in E$. For $\Lambda \subseteq G$ and $\tilde{E} \subseteq E$ we define $\tilde{E}|_\Lambda = \{[x, y] \in \tilde{E} \mid x, y \in \Lambda\}$. Two sets $\Lambda, \Lambda' \subseteq G$ are called equivalent in a given configuration $\omega \in \Omega$ (and we write $\Lambda \stackrel{\omega}{\sim} \Lambda'$) if there exists $x \in G$ with $\Lambda x = \Lambda'$ and $\omega_{v,s} = \omega_{vx,s}$ for all $e \in E|_\Lambda$, $s \in S$. Two vertices $x, y \in G$ are said to be connected in the configuration ω if there exists an active path in Γ_ω connecting x and y , i.e. a sequence of vertices $x = v_0, v_1, \dots, v_n = y \in G$ such that $[v_{i-1}, v_i] \in E$ and $X_{v_{i-1}, v_i v_{i-1}^{-1}}(\omega) = X_{v_i, v_{i-1} v_i^{-1}}(\omega) = 1$ for all $i = 1, \dots, n$. In this situation we write $x \stackrel{\omega}{\leftrightarrow} y$. For $x \in G$ denote by

$$C_x(\omega) = \{y \in G \mid x \stackrel{\omega}{\leftrightarrow} y\}$$

the cluster of x in the configuration ω . For a given subset $\Lambda \subseteq G$, we define similarly

$$C_x^\Lambda(\omega) = \{y \in G \mid x \stackrel{\omega, \Lambda}{\leftrightarrow} y\} \subseteq \Lambda,$$

where $x \stackrel{\omega, \Lambda}{\leftrightarrow} y$ means that there exists an active path in Γ_ω such that all the vertices in the path are elements of Λ . Note that $x \stackrel{\omega, \Lambda}{\leftrightarrow} y$ implies $C_x^\Lambda(\omega) = C_y^\Lambda(\omega)$.

Now we define a function which counts clusters in a given set up to a certain size. For given $\omega \in \Omega$ and $\Lambda \in \mathcal{F}(G)$ we define $F_\omega(\Lambda) : \mathbb{R} \rightarrow \mathbb{N}_0$ by

$$F_\omega(\Lambda)(m) := \# \{C_x^\Lambda(\omega) \mid x \in \Lambda, |C_x^\Lambda(\omega)| \leq m\}. \quad (10.1)$$

For each $\Lambda \in \mathcal{F}(G)$ and $\omega \in \Omega$ the function $F_\omega(\Lambda)$ is an element of $\mathcal{B}(\mathbb{R})$, which is the space of all bounded and right-continuous functions, equipped with the supremum norm. We are

interested in the limit $F_\omega(\Lambda_n)/K_\omega(\Lambda_n)$ as $n \rightarrow \infty$, where for some $\Lambda \in \mathcal{F}(G)$ the expression $K_\omega(\Lambda)$ denotes the number of clusters in Λ , i.e.

$$K_\omega(\Lambda) := \sharp \{C_x^\Lambda(\omega) \mid x \in \Lambda\} \quad (10.2)$$

Let us now state the main result of this section.

Theorem 10.2. *Let G be an amenable group, (Λ_n) a tempered Følner sequence and $(\Omega, \mathcal{Z}, \mathbb{P})$ the same probability space as above. Furthermore, let the function $F_\omega : \mathcal{F}(G) \rightarrow \mathcal{B}(\mathbb{R})$ be given as in (10.1) and $K_\omega(\Lambda_n)$ as in (10.2). Then there exists a set $\tilde{\Omega} \subseteq \Omega$ of full measure and a distribution function $\Phi \in \mathcal{B}(\mathbb{R})$ associated to a probability measure, such that*

$$\lim_{n \rightarrow \infty} \left\| \frac{F_\omega(\Lambda_n)}{K_\omega(\Lambda_n)} - \Phi \right\| = 0$$

holds for all $\omega \in \tilde{\Omega}$, where $\|\cdot\|$ denotes the supremum norm in $\mathcal{B}(\mathbb{R})$.

By definition it is clear that $F_\omega(\Lambda) = F_\omega(\Lambda')$ whenever $\Lambda \stackrel{\omega}{\sim} \Lambda'$. Furthermore, if $\Lambda \subseteq G$ and $\omega, \omega' \in \Omega$ are such that

$$\sharp(E_\omega|_\Lambda \triangle E_{\omega'}|_\Lambda) = \sharp((E_\omega|_\Lambda \setminus E_{\omega'}|_\Lambda) \cup (E_{\omega'}|_\Lambda \setminus E_\omega|_\Lambda)) = 1,$$

then

$$|F_\omega(\Lambda)(m) - F_{\omega'}(\Lambda)(m)| \leq 2.$$

for all $m \in \mathbb{R}$. This is clear since the change of one edge from active to non-active (respectively from non-active to active) will perturb the number of clusters in Λ of a fixed size at most by 2. Generalizing this idea leads to the following lemma.

Lemma 10.3. *Let $\omega \in \Omega$, $\Lambda \subseteq G$ finite and disjoint $\Lambda_1, \dots, \Lambda_k \subseteq \Lambda$ with $\bigcup_{i=1}^k \Lambda_i = \Lambda$ be given. Then*

$$\left| F_\omega(\Lambda)(m) - \sum_{i=1}^k F_\omega(\Lambda_i)(m) \right| \leq 2|S| \sum_{i=1}^k |\partial^1(\Lambda_i)|$$

holds for all $m \in \mathbb{R}$.

Proof. Fix $m \in \mathbb{R}$ and let $B := \{[x, y] \in E|_\Lambda \mid x \in \Lambda_i, y \in \Lambda_j \text{ for some } i \neq j\}$ and $\omega' \in \Omega$ be such that $e \notin E_{\omega'}$ for all $e \in B$ and $e' \in E_{\omega'} \setminus B$ if and only if $e' \in E_\omega \setminus B$. Using the preliminary considerations one can show inductively that

$$|F_\omega(\Lambda)(m) - F_{\omega'}(\Lambda)(m)| \leq 2 \cdot \sharp(B).$$

Furthermore, as each edge in B connects a vertex in $\partial^1(\Lambda_i)$ with a vertex in $\partial^1(\Lambda_j)$ for some distinct $i, j \in \{1, \dots, k\}$ and each vertex in Γ is adjacent to exactly $|S|$ vertices, we estimate the number of elements in B by $\sharp(B) \leq |S| \sum_{i=1}^k |\partial^1(\Lambda_i)|$. It remains to show $F_{\omega'}(\Lambda)(m) = \sum_{i=1}^k F_\omega(\Lambda_i)(m)$. Here we use a decoupling argument. In the configuration ω' we have for arbitrary $i \in \{1, \dots, k\}$ and $x \in \Lambda_i$ that $C_x^\Lambda(\omega') \cap \Lambda_j = \emptyset$ for all $j \neq i$ as all the edges in B are non-active in ω' . Therefore,

$$F_{\omega'}(\Lambda)(m) = \sum_{i=1}^k F_{\omega'}(\Lambda_i)(m) = \sum_{i=1}^k F_\omega(\Lambda_i)(m)$$

holds, where the last equality follows from the fact that $e \in E_{\omega'} \Leftrightarrow e \in E_\omega$ for all $e \in E|_{\Lambda_i}$, $i \in \{1, \dots, k\}$. The claim follows. \blacksquare

Note that by setting $\mathcal{A} := \{0, 1\}^S$ and $\mathcal{C}_\omega : V \rightarrow \mathcal{A}$, $\mathcal{C}_\omega(v) := (\omega_{v,s})_{s \in S}$, each $\omega \in \Omega$ may be interpreted as coloring of the vertices. It is obvious that for each $\omega \in \Omega$ the function $F_\omega(\cdot)$ is \mathcal{C}_ω -invariant, i.e. for any $Q, U \in \mathcal{F}(G)$ the equivalence of the patterns $\mathcal{C}_\omega|_Q$ and $\mathcal{C}_\omega|_U$ implies $F_\omega(Q) = F_\omega(U)$, cf. Definition 5.2.

In order to apply Theorem 5.5 it remains to prove the existence of the frequencies. Here we use the notation

$$\sharp_P(\mathcal{C}_\omega|_\Lambda) := \sharp \left\{ x \in G \mid \tilde{\Lambda}x \subseteq \Lambda, P(y) = \mathcal{C}_\omega(yx) \text{ for all } y \in \tilde{\Lambda} \right\},$$

where $\Lambda, \tilde{\Lambda} \in \mathcal{F}(G)$, $P : \tilde{\Lambda} \rightarrow \mathcal{A}$ and $\omega \in \Omega$.

Lemma 10.4. *There exists a set $\tilde{\Omega} \subseteq \Omega$ of full measure such that for all $\omega \in \tilde{\Omega}$ and all $P : \Lambda \rightarrow \mathcal{A}$, $\Lambda \in \mathcal{F}(G)$ the limit*

$$\nu_P := \lim_{n \rightarrow \infty} \frac{\sharp_P(\mathcal{C}_\omega|_{\Lambda_n})}{|\Lambda_n|} = \prod_{v \in \Lambda} \prod_{s \in S} (p_s \delta_{1, (P(v))_s} + (1 - p_s) \delta_{0, (P(v))_s})$$

exists and does not depend on ω . Here $\delta_{i,j}$ denotes the Kronecker delta.

This lemma is basically an application of Lindenstrauss' pointwise ergodic theorem (cf. [Lin01]) in the special quoted in Theorem 6.1. The proof of Lemma 10.4 follows the lines of the proof of Theorem 6.2 using the mapping

$$T : G \times \Omega \rightarrow \Omega, \quad (g, \omega) \mapsto T_g \omega, \quad \text{where} \quad (T_g \omega)_{v,s} = \omega_{vg,s}. \quad (10.3)$$

Note that T is an ergodic measure preserving left action on the probability space.

The number of clusters per vertex is defined by

$$\kappa := \mathbb{E}(|C_{\text{id}}|^{-1}) = \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}(|C_{\text{id}}| = n).$$

Note that $\kappa \in (0, 1]$ as we assumed $p \in [0, 1]^S$.

Lemma 10.5. *Let G be an amenable group, (Λ_n) a tempered Følner sequence and $(\Omega, \mathcal{Z}, \mathbb{P})$ the same probability space as above. Then*

$$\lim_{n \rightarrow \infty} \frac{K_\omega(\Lambda_n)}{|\Lambda_n|} = \kappa$$

holds for \mathbb{P} -almost all ω .

Proof. We adapt the proof of Theorem 4.2 in [Gri99] to the case of amenable groups. For $x \in G$, $\omega \in \Omega$ and $n \in \mathbb{N}$ set

$$\gamma_x(\omega) := \begin{cases} |C_x(\omega)|^{-1}, & \text{if } |C_x(\omega)| < \infty \\ 0, & \text{else} \end{cases} \quad \text{and} \quad \gamma_x^{(n)}(\omega) := |C_x^{\Lambda_n}(\omega)|^{-1}.$$

As $|C_x(\omega)| \geq |C_x^{\Lambda_n}(\omega)|$, this obviously yields

$$\gamma_x(\omega) \leq \gamma_x^{(n)}(\omega). \quad (10.4)$$

Furthermore, since for each $y \in \Lambda_n$, one has

$$\sum_{x \in C_y^{\Lambda_n}(\omega)} \gamma_x^{(n)}(\omega) = |C_y^{\Lambda_n}(\omega)| \gamma_y^{(n)}(\omega) = 1,$$

we obtain

$$K_\omega(\Lambda_n) = \sum_{x \in \Lambda_n} \gamma_x^{(n)}(\omega).$$

This, together with (10.4), gives

$$\frac{1}{|\Lambda_n|} K_\omega(\Lambda_n) \geq \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \gamma_x(\omega).$$

Note that with $T : \Omega \rightarrow \Omega$ given as in (10.3) we have $\gamma_x(\omega) = \gamma_{\text{id}}(T_x \omega)$. Therefore, with the Lindenstrauss ergodic theorem, we arrive at

$$\frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \gamma_x(\omega) = \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \gamma_{\text{id}}(T_x \omega) \rightarrow \mathbb{E}(\gamma_{\text{id}}) \quad \text{as } n \rightarrow \infty$$

for \mathbb{P} -almost all $\omega \in \Omega$. This proves the following lower bound

$$\liminf_{n \rightarrow \infty} \left(\frac{1}{|\Lambda_n|} K_\omega(\Lambda_n) \right) \geq \mathbb{E}(\gamma_{\text{id}}) = \kappa \quad \mathbb{P} - \text{a.s.}$$

It remains to show that κ is also an upper bound for $\limsup_n (K_\omega(\Lambda_n)/|\Lambda_n|)$. To this end, we estimate

$$\begin{aligned} \sum_{x \in \Lambda_n} \gamma_x^{(n)}(\omega) &= \sum_{x \in \Lambda_n} \gamma_x(\omega) + \sum_{x \in \Lambda_n} (\gamma_x^{(n)}(\omega) - \gamma_x(\omega)) \\ &\leq \sum_{x \in \Lambda_n} \gamma_x(\omega) + \sum_{\substack{x \in \Lambda_n \\ C_x(\omega) \cap \partial^1(\Lambda_n) \neq \emptyset}} \gamma_x^{(n)}(\omega). \end{aligned}$$

Here we used that $\gamma_x^{(n)}(\omega) = \gamma_x(\omega)$ whenever $C_x(\omega)$ is a subset of Λ_n . Note that the second sum is nothing but the number of all clusters $C_x(\omega)$, $x \in \Lambda_n$ which have a non-empty intersection with the boundary $\partial^1(\Lambda_n)$. This number is bounded from above by $|\partial^1(\Lambda_n)|$. In light of that,

$$\frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \gamma_x^{(n)}(\omega) \leq \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \gamma_x(\omega) + \frac{|\partial^1(\Lambda_n)|}{|\Lambda_n|}.$$

We conclude that

$$\limsup_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \gamma_x^{(n)}(\omega) \leq \limsup_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \gamma_x(\omega) = \kappa \quad \mathbb{P} - \text{a.s.},$$

since (Λ_n) is a Følner sequence. ■

We are now in position to prove Theorem 10.2.

Proof of Theorem 10.2. Let $\tilde{\Omega} \subseteq \Omega$ be given as in Lemma 10.4 and fix some $\omega \in \tilde{\Omega}$. We have already seen that $F_\omega : \mathcal{F} \rightarrow \mathcal{B}(\mathbb{R})$ is \mathcal{C}_ω -invariant and in Lemma 10.3 we proved that F_ω is almost additive with boundary term $2|S||\partial^1(\cdot)|$. Therefore Theorem 5.5 yields that there exists a function $F : \mathcal{F}(G) \rightarrow \mathcal{B}(\mathbb{R})$ with

$$\lim_{n \rightarrow \infty} \left\| \frac{F_\omega(\Lambda_n)}{|\Lambda_n|} - F \right\| = 0$$

in the supremum norm. Thus, we get by Lemma 10.5,

$$\lim_{n \rightarrow \infty} \left\| \frac{F_\omega(\Lambda_n)}{K_\omega(\Lambda_n)} - \Phi \right\| = \lim_{n \rightarrow \infty} \left\| \frac{F_\omega(\Lambda_n)}{|\Lambda_n|} \frac{|\Lambda_n|}{K_\omega(\Lambda_n)} - \Phi \right\| = 0,$$

where $\Phi = \frac{1}{\kappa} F$. It remains to prove that Φ is a distribution function of a probability measure. To show this, we note first that for $m \leq \tilde{m}$ one has

$$\Phi(\tilde{m}) - \Phi(m) = \lim_{n \rightarrow \infty} \frac{F_\omega(\Lambda_n)(\tilde{m}) - F_\omega(\Lambda_n)(m)}{\kappa |\Lambda_n|} \geq 0$$

since each $F_\omega(\Lambda_n)$ is monotonically non-decreasing. To obtain the right-continuity of Φ , one needs to show $\Phi(m) = \lim_{\tilde{m} \searrow m} \Phi(\tilde{m})$ for all $m \in \mathbb{R}$. This is clear from

$$\lim_{\tilde{m} \searrow m} \Phi(\tilde{m}) = \lim_{\tilde{m} \searrow m} \lim_{n \rightarrow \infty} \frac{F_\omega(\Lambda_n)(\tilde{m})}{\kappa |\Lambda_n|} = \lim_{n \rightarrow \infty} \lim_{\tilde{m} \searrow m} \frac{F_\omega(\Lambda_n)(\tilde{m})}{\kappa |\Lambda_n|} = \lim_{n \rightarrow \infty} \frac{F_\omega(\Lambda_n)(m)}{\kappa |\Lambda_n|} = \Phi(m),$$

where we used the uniform convergence to interchange the limits as well as the right-continuity of the functions $F_\omega(\Lambda_n)$. Applying again the uniform convergence of the approximants gives

$$\lim_{m \rightarrow \infty} F(m) = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{F_\omega(\Lambda_n)(m)}{|\Lambda_n|} = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \frac{F_\omega(\Lambda_n)(m)}{|\Lambda_n|} = \lim_{n \rightarrow \infty} \frac{K_\omega(\Lambda_n)}{|\Lambda_n|} = \kappa,$$

which implies $\lim_{m \rightarrow \infty} \Phi(m) = 1$. The fact that $\lim_{m \rightarrow -\infty} \Phi(m) = 0$ is obvious. \blacksquare

Corollary 10.6. *Let G be an amenable group, (Λ_n) a tempered Følner sequence and $(\Omega, \mathcal{Z}, \mathbb{P})$ the same probability space as above. Then there exists a set of full measure $\tilde{\Omega} \subseteq \Omega$ such that for each $m \in \mathbb{N}$ and $\omega \in \tilde{\Omega}$ the densities of clusters of size m defined as the limits*

$$c_m := \lim_{n \rightarrow \infty} \frac{1}{K_\omega(\Lambda_n)} \# \{C_x^{\Lambda_n} \mid x \in \Lambda_n, |C_x^{\Lambda_n}(\omega)| = m\},$$

$$d_m := \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \# \{x \in \Lambda_n \mid |C_x(\omega)| = m\} \quad \text{and} \quad d_\infty := \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \# \{x \in \Lambda_n \mid |C_x(\omega)| = \infty\}$$

exist and do not depend on (Λ_n) and $\omega \in \tilde{\Omega}$. Furthermore, the convergence of c_m is uniform in m and $\sum_{m \in \mathbb{N}} c_m = 1$ holds.

Remark 10.7. To define the densities c_m , one counts clusters of size m in a certain set and normalizes by the number of all clusters in this set. In contrast to that, to define d_m one counts the vertices in clusters of size m in a certain set and normalizes by the number of all vertices in this set. The distribution function F given by Theorem 10.2 is associated with the densities c_m . Using the uniform convergence of the approximants one can show the uniform existence of the c_m and the fact that the c_m sum up to one. One can also define an associated

distribution function $G_\omega(\Lambda) : \mathbb{R} \rightarrow \mathbb{N}_0$ for the densities d_m by setting for given $\omega \in \Omega$ and $\Lambda \in \mathcal{F}(G)$

$$G_\omega(\Lambda)(m) := \sharp \{x \in \Lambda \mid |C_x^\Lambda(\omega)| \leq m\}.$$

However, it is not possible to prove an adapted version of Lemma 10.3 for the functions $G_\omega(\Lambda)$ and hence one cannot apply the ergodic theorem to obtain uniform convergence. This causes the qualitative difference in the results for the densities c_m and d_m .

Proof. Let us begin with the existence of the density d_∞ . To do so we use $|C_x(\omega)| = |C_{\text{id}}(T_x\omega)|$ to obtain

$$\begin{aligned} \frac{\sharp \{x \in \Lambda_n \mid |C_x(\omega)| = \infty\}}{|\Lambda_n|} &= \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \mathbf{1}_{\{x \in \Lambda_n \mid |C_x(\omega)| = \infty\}}(x) \\ &= \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \mathbf{1}_{\{x \in \Lambda_n \mid |C_{\text{id}}(T_x\omega)| = \infty\}}(x). \end{aligned}$$

The application of Theorem 6.1 with $\varphi : \Omega \rightarrow \{0, 1\}$, $\varphi(\omega) := \mathbf{1}_{\{|C_{\text{id}}(\omega)| = \infty\}}(\omega)$ shows that there exists a set $\Omega' \subseteq \Omega$ of full measure such that the following limits exist for all $\omega \in \Omega'$

$$\lim_{n \rightarrow \infty} \frac{\sharp \{x \in \Lambda_n \mid |C_x(\omega)| = \infty\}}{|\Lambda_n|} = \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \varphi(T_x\omega) = \mathbb{E}\varphi = \mathbb{P}(|C_{\text{id}}| = \infty) =: d_\infty.$$

Now let $\Omega'' \subseteq \Omega$ be the set of full measure such that Theorem 10.2 holds for all $\omega \in \Omega''$ and set $\tilde{\Omega} := \Omega' \cap \Omega''$. From now on we fix an element $\omega \in \tilde{\Omega}$. For each $\Lambda \in \mathcal{F}(G)$, set

$$f_\omega(\Lambda) : \mathbb{R} \rightarrow \mathbb{N}_0, \quad m \mapsto f_\omega(\Lambda)(m) := \sharp \{C_x^\Lambda \mid x \in \Lambda, |C_x^\Lambda| = \lfloor m \rfloor\},$$

$$g_\omega(\Lambda) : \mathbb{R} \rightarrow \mathbb{N}_0, \quad m \mapsto g_\omega(\Lambda)(m) := \sharp \{x \in \Lambda \mid |C_x^\Lambda| = \lfloor m \rfloor\}$$

and $f, g : \mathbb{R} \rightarrow \mathbb{R}$ by $f(m) := F(m) - F(m-1)$ and $g(m) := mf(m)$, where F is the limit function given by $F = \lim_{n \rightarrow \infty} F_\omega(\Lambda_n)/|\Lambda_n|$, see the proof of Theorem 10.2. Let $\varepsilon > 0$ be given. As $F_\omega(\Lambda_n)/|\Lambda_n|$ converges uniformly to F , there is an $n_0 \in \mathbb{N}$ such that

$$\left| \frac{F_\omega(\Lambda_n)(m)}{|\Lambda_n|} - F(m) \right| \leq \varepsilon$$

for all $m \in \mathbb{R}$ ($n > n_0$). Therefore

$$\begin{aligned} \left| \frac{f_\omega(\Lambda_n)(m)}{|\Lambda_n|} - f(m) \right| &= \left| \frac{F_\omega(\Lambda_n)(m) - F_\omega(\Lambda_n)(m-1)}{|\Lambda_n|} - (F(m) - F(m-1)) \right| \\ &\leq \left| \frac{F_\omega(\Lambda_n)(m)}{|\Lambda_n|} - F(m) \right| + \left| \frac{F_\omega(\Lambda_n)(m-1)}{|\Lambda_n|} - F(m-1) \right| \leq 2\varepsilon \end{aligned}$$

holds for all $m \in \mathbb{R}$ and all $n > n_0$. This shows that $f_\omega(\Lambda_n)/|\Lambda_n|$ converges uniformly to f . For $m \in \mathbb{N}$, $\varepsilon > 0$ and n_0 chosen as above, consider

$$\left| \frac{g_\omega(\Lambda_n)(m)}{|\Lambda_n|} - g(m) \right| = m \left| \frac{f_\omega(\Lambda_n)(m)}{|\Lambda_n|} - f(m) \right| \leq 2\varepsilon m$$

for all $n \geq n_0$. Since

$$\lim_{n \rightarrow \infty} \left| \frac{\sharp \{x \in \Lambda_n \mid |C_x(\omega)| = m\}}{|\Lambda_n|} - g(m) \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{\# \{x \in \Lambda_n \mid |C_x(\omega)| = m\} - \# \{x \in \Lambda_n \mid |C_x^{\Lambda_n}(\omega)| = m\}}{|\Lambda_n|} \right| \leq \lim_{n \rightarrow \infty} \frac{|\partial^m(\Lambda_n)|}{|\Lambda_n|} = 0,$$

this proves the existence of the densities $d_m = g(m)$ for each $m \in \mathbb{N}$.

Further, given $\varepsilon > 0$, choose $n_1 \in \mathbb{N}$ such that

$$\left| \frac{|\Lambda_n|}{K_\omega(\Lambda_n)} - \frac{1}{\kappa} \right| \leq \frac{\varepsilon}{2} \quad \text{and} \quad \left| \frac{f_\omega(\Lambda_n)(m)}{|\Lambda_n|} - f(m) \right| \leq \frac{\kappa \varepsilon}{2}$$

for all $m \in \mathbb{R}$ and $n \geq n_1$. Then we have

$$\left| \frac{f_\omega(\Lambda_n)(m)}{K_\omega(\Lambda_n)} - \frac{1}{\kappa} f(m) \right| \leq \left| \frac{f_\omega(\Lambda_n)(m)}{|\Lambda_n|} \frac{|\Lambda_n|}{K_\omega(\Lambda_n)} - \frac{f_\omega(\Lambda_n)(m)}{\kappa |\Lambda_n|} \right| + \left| \frac{f_\omega(\Lambda_n)(m)}{\kappa |\Lambda_n|} - \frac{1}{\kappa} f(m) \right| \leq \varepsilon$$

for all $m \in \mathbb{R}$ and $n \geq n_1$, which proves the uniform existence of c_m , where $c_m = \frac{1}{\kappa} f(m)$. Furthermore, we have indeed

$$\sum_{m \in \mathbb{N}} c_m = \frac{1}{\kappa} \sum_{m \in \mathbb{N}} f(m) = \frac{1}{\kappa} \lim_{m \rightarrow \infty} F(m) = \frac{1}{\kappa} \kappa = 1.$$

■

10.2 Continuous dependence

In this subsection we study the dependence of the limit distribution Φ given by Theorem 10.2 on the percolation parameter $p = (p_s)_{s \in S} \in [0, 1]^S$. We measure distances in $[0, 1]^S$ with the usual ℓ^2 -metric. To emphasize the dependence on p we use the notation \mathbb{P}_p , \mathbb{E}_p , $\kappa(p)$ and $\Phi(p)$ instead of \mathbb{P} , \mathbb{E} , κ and Φ . For each $\Lambda \in \mathcal{F}(G)$ we define

$$\bar{F}(\Lambda) : [0, 1]^S \rightarrow \mathcal{B}(\mathbb{R}), \quad p \mapsto \bar{F}_p(\Lambda), \quad (10.5)$$

where

$$\bar{F}_p(\Lambda)(m) = \mathbb{E}_p(F_\omega(\Lambda)(m)) = \mathbb{E}_p(\# \{C_x^\Lambda(\omega) \mid x \in \Lambda, |C_x^\Lambda(\omega)| \leq m\})$$

for all $m \in \mathbb{R}$. Note that for fixed p and Λ the function $\bar{F}_p(\Lambda)$ is constant on each interval $[k, k+1)$ for $k \in \mathbb{Z}$ and hence in $\mathcal{B}(\mathbb{R})$. Furthermore, we have for arbitrary $m \in \mathbb{R}$ and $\Lambda \in \mathcal{F}(G)$

$$\begin{aligned} \bar{F}_p(\Lambda)(m) &= \sum_{k=1}^{|\Lambda|} k \cdot \mathbb{P}_p(\{\omega \in \Omega \mid F_\omega(\Lambda)(m) = k\}) \\ &= \sum_{k=1}^{|\Lambda|} k \sum_{a \in \{0,1\}^{\Lambda \times S}} \mathbb{P}_p(Z(a)) \mathbf{1}_{\{F_\omega(\Lambda)(m)=k \text{ for some } \omega \in Z(a)\}}(a), \end{aligned}$$

where $Z(a) = \{\omega \in \Omega \mid \omega_{v,s} = a_{v,s} \text{ for all } v \in \Lambda, s \in S\}$ for $a = (a_{v,s})_{v \in \Lambda, s \in S} \in \{0,1\}^{\Lambda \times S}$. Here we used that for fixed a, m and Λ one has $F_\omega(\Lambda)(m) = k$ for all $\omega \in Z(a)$ if and only if $F_\omega(\Lambda)(m) = k$ for some $\omega \in Z(a)$. Obviously we have

$$\mathbb{P}_p(Z(a)) = \prod_{v \in \Lambda} \prod_{s \in S} (p_s \delta_{1,a_{v,s}} + (1 - p_s) \delta_{0,a_{v,s}}),$$

where $\delta_{i,j}$ is the Kronecker delta. This shows that $p \mapsto \mathbb{P}_p(Z(a))$ is a multivariate polynomial in p_s , $s \in S$ and hence, it is continuous in $p = (p_s)_{s \in S}$. Canonically, we use the ℓ^2 -norm and the induced metric in $[0, 1]^S$. Now it is clear that for each $m \in \mathbb{R}$ and $\Lambda \in \mathcal{F}(G)$ the mapping $p \mapsto \bar{F}_p(\Lambda)(m)$ is continuous as well. Therefore given $\varepsilon > 0$, $\Lambda \in \mathcal{F}(G)$ and $p \in [0, 1]^S$ we can find $\delta > 0$ such that $|\bar{F}_p(\Lambda)(m) - \bar{F}_{p'}(\Lambda)(m)| \leq \varepsilon$ whenever $p' \in [0, 1]^S$ with $\|p - p'\|_2 \leq \delta$ for all $m \in \{0, 1, \dots, |\Lambda|\}$. Hence, for these p' we have $\|\bar{F}_p(\Lambda) - \bar{F}_{p'}(\Lambda)\| \leq \varepsilon$, which proves the continuity of the function $p \mapsto \bar{F}_p(\Lambda)$. This shows that $\Lambda \mapsto \bar{F}(\Lambda)$ maps elements of $\mathcal{F}(G)$ into the space of all continuous functions mapping from $[0, 1]^S$ to $\mathcal{B}(\mathbb{R})$. We write

$$\mathcal{C}([0, 1]^S, \mathcal{B}(\mathbb{R})) = \{\varphi : [0, 1]^S \rightarrow \mathcal{B}(\mathbb{R}) \mid \varphi \text{ is continuous}\}$$

and we equip this space with the supremum norm

$$\|\varphi\|_\infty = \sup_{p \in [0, 1]^S} \|\varphi(p)\| = \sup_{p \in [0, 1]^S} \sup_{m \in \mathbb{R}} |\varphi(p)(m)|.$$

Note that with this norm $\mathcal{C}([0, 1]^S, \mathcal{B}(\mathbb{R}))$ is a Banach space. Our next goal is to study the limit of $\bar{F}(\Lambda_n)/|\Lambda_n|$ for some Følner sequence (Λ_n) .

The following result has been shown by Grimmett for \mathbb{Z}^d in [Gri76]. We generalize this idea to the case of amenable groups.

Lemma 10.8. *The mapping $\kappa : [0, 1]^S \rightarrow (0, 1]$, $p \mapsto \kappa(p) = \mathbb{E}_p(|C_{\text{id}}|^{-1})$ is continuous.*

Proof. Let $(\Lambda_n)_{n \in \mathbb{N}}$ be a tempered Følner sequence. By Lemma 10.5, for each $p \in [0, 1]^S$ there is a set Ω_p with $\mathbb{P}_p(\Omega_p) = 1$ and $\kappa(p) = \lim_{n \rightarrow \infty} K_\omega(\Lambda_n)/|\Lambda_n|$ for all $\omega \in \Omega_p$, which gives

$$\limsup_{n \rightarrow \infty} \frac{K_\omega(\Lambda_n)}{|\Lambda_n|} = \kappa(p) = \liminf_{n \rightarrow \infty} \frac{K_\omega(\Lambda_n)}{|\Lambda_n|} \quad \mathbb{P}_p\text{-a.s.}$$

Fatou's Lemma implies

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E}_p(K_\omega(\Lambda_n))}{|\Lambda_n|} \leq \kappa(p) \leq \liminf_{n \rightarrow \infty} \frac{\mathbb{E}_p(K_\omega(\Lambda_n))}{|\Lambda_n|}$$

and hence $\kappa(p) = \lim_{n \rightarrow \infty} \mathbb{E}_p(K_\omega(\Lambda_n))/|\Lambda_n|$. Now we show that this limit exists even uniformly in p and that the limit function is continuous. To do so, set

$$H : \mathcal{F}(G) \rightarrow \mathcal{B}([0, 1]^S), \quad H(\Lambda)(p) := \mathbb{E}_p(K_\omega(\Lambda)) = \mathbb{E}_p(\#\{C_x^\Lambda(\omega) \mid x \in \Lambda\}),$$

where $\mathcal{B}([0, 1]^S)$ denotes the Banach-space of bounded and continuous functions mapping from $[0, 1]^S$ to \mathbb{R} , equipped with the supremum norm. Note that the mapping $p \mapsto \mathbb{E}_p(K_\omega(\Lambda))$ is in $\mathcal{B}([0, 1]^S)$ since

$$\mathbb{E}_p(K_\omega(\Lambda)) = \sum_{k=1}^{|\Lambda|} k \cdot \mathbb{P}_p(\{\omega \in \Omega \mid K_\omega(\Lambda) = k\})$$

and $\mathbb{P}_p(\{\omega \in \Omega \mid K_\omega(\Lambda) = k\})$ is a polynomial in p_s , $s \in S$. By the translation invariance of the measure, we have $H(\Lambda) = H(\Lambda z)$ for all $z \in G$ and $\Lambda \in \mathcal{F}(G)$, hence H is \mathcal{C}_ω -invariant for each $\omega \in \Omega$. Moreover, we claim that for disjoint $\Lambda_1, \dots, \Lambda_k \in \mathcal{F}(G)$, we have

$$\left\| H(\Lambda) - \sum_{i=1}^k H(\Lambda_i) \right\| \leq 2|S| \sum_{i=1}^k |\partial(\Lambda_i)|, \quad (10.6)$$

where $\Lambda = \bigcup_{i=1}^k \Lambda_i$. This is true since for any $p \in [0, 1)^S$,

$$\left| \mathbb{E}_p(K_\omega(\Lambda)) - \sum_{i=1}^k \mathbb{E}_p(K_\omega(\Lambda_i)) \right| \leq \mathbb{E}_p \left(\left| K_\omega(\Lambda) - \sum_{i=1}^k K_\omega(\Lambda_i) \right| \right) \leq 2|S| \sum_{i=1}^k |\partial(\Lambda_i)|.$$

Note that the latter inequality holds since $K_\omega(\Lambda) = F_\omega(\Lambda)(|\Lambda|)$ and by Lemma 10.3. Therefore, Theorem 5.5 yields that the functions $H(\Lambda_n)/|\Lambda_n|$ converge uniformly to $\kappa \in \mathcal{B}([0, 1)^S)$ as n tends to infinity. \blacksquare

The main result of this subsection is the following.

Theorem 10.9. *Let (Λ_n) be a tempered Følner sequence and let for each $p \in [0, 1)^S$ the function $\Phi_p \in \mathcal{B}(\mathbb{R})$ be the limit given by Theorem 10.2. Then the function $\Psi : [0, 1)^S \rightarrow \mathcal{B}(\mathbb{R})$, $p \mapsto \Phi_p$ is continuous.*

Proof. Let $\bar{F} : \mathcal{F}(G) \rightarrow \mathcal{C}([0, 1)^S, \mathcal{B}(\mathbb{R}))$ be given as in (10.5). Furthermore let $\Lambda_1, \dots, \Lambda_k \in \mathcal{F}(G)$ be disjoint and set $\Lambda = \bigcup_{i=1}^k \Lambda_i$. Then for any $p \in [0, 1)^S$ and $m \in \mathbb{R}$ we have

$$\left| \bar{F}_p(\Lambda)(m) - \sum_{i=1}^k \bar{F}_p(\Lambda_i)(m) \right| \leq \mathbb{E}_p \left(\left| F_\omega(\Lambda)(m) - \sum_{i=1}^k F_\omega(\Lambda_i)(m) \right| \right) \leq 2|S| \sum_{i=1}^k \partial(\Lambda_i),$$

where the last inequality follows from Lemma 10.3. Therefore,

$$\left\| \bar{F}(\Lambda) - \sum_{i=1}^k \bar{F}(\Lambda_i) \right\| = \sup_{p \in [0, 1)^S} \sup_{m \in \mathbb{R}} \left| F_p(\Lambda)(m) - \sum_{i=1}^k \bar{F}_p(\Lambda_i)(m) \right| \leq 2|S| \sum_{i=1}^k \partial(\Lambda_i).$$

Besides, we have $\bar{F}(\Lambda) = \bar{F}(\Lambda x)$ for arbitrary $\Lambda \in \mathcal{F}(G)$ and $x \in G$ which allows us to apply Theorem 5.5. This proves the existence of a function $\tilde{F} \in \mathcal{C}([0, 1)^S, \mathcal{B}(\mathbb{R}))$ with

$$\lim_{n \rightarrow \infty} \left\| \frac{\bar{F}(\Lambda_n)}{|\Lambda_n|} - \tilde{F} \right\|_\infty = 0.$$

We claim that $\Phi_p = \tilde{F}(p)/\kappa(p)$. If this holds, the proof is completed since $\kappa : [0, 1)^S \rightarrow \mathbb{R}$ is continuous by Lemma 10.8. To prove the claim, note first that

$$\begin{aligned} \left\| \frac{F_\omega(\Lambda_n)}{|\Lambda_n|} - \kappa(p)\Phi_p \right\| &\leq \left\| \frac{F_\omega(\Lambda_n)}{|\Lambda_n|} - \frac{K_\omega(\Lambda_n)}{|\Lambda_n|}\Phi_p \right\| + \left\| \frac{K_\omega(\Lambda_n)}{|\Lambda_n|}\Phi_p - \kappa(p)\Phi_p \right\| \\ &\leq \left| \frac{K_\omega(\Lambda_n)}{|\Lambda_n|} \right| \cdot \left\| \frac{F_\omega(\Lambda_n)}{K_\omega(\Lambda_n)} - \Phi_p \right\| + \left| \frac{K_\omega(\Lambda_n)}{|\Lambda_n|} - \kappa(p) \right| \cdot \|\Phi_p\| \end{aligned}$$

implies that

$$\lim_{n \rightarrow \infty} \left\| \frac{F_\omega(\Lambda_n)}{|\Lambda_n|} - \kappa(p)\Phi_p \right\| = 0 \quad \mathbb{P}_p\text{-a.s.}$$

Thus, by the Lebesgue convergence theorem, we obtain

$$\begin{aligned} 0 &= \mathbb{E}_p \left(\lim_{n \rightarrow \infty} \left\| \frac{F_\omega(\Lambda_n)}{|\Lambda_n|} - \kappa(p)\Phi_p \right\| \right) \\ &= \lim_{n \rightarrow \infty} \mathbb{E}_p \left(\left\| \frac{F_\omega(\Lambda_n)}{|\Lambda_n|} - \kappa(p)\Phi_p \right\| \right) \end{aligned}$$

$$\geq \limsup_{n \rightarrow \infty} \left\| \frac{\mathbb{E}_p(F_\omega(\Lambda_n))}{|\Lambda_n|} - \kappa(p)\Phi_p \right\|$$

for each $p \in [0, 1)^S$ and hence

$$\lim_{n \rightarrow \infty} \left\| \frac{\bar{F}_p(\Lambda_n)}{|\Lambda_n|} - \kappa(p)\Phi_p \right\| = 0.$$

This finishes the proof of the claim. ■

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